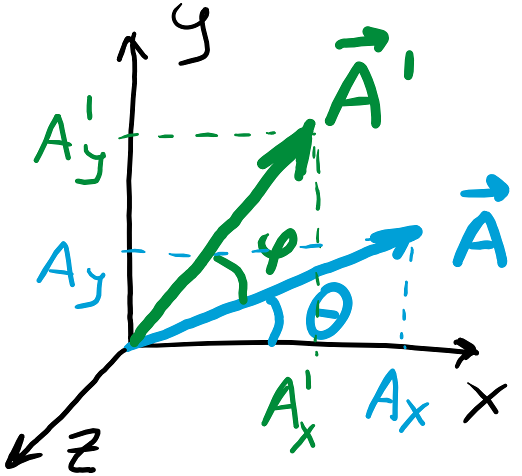
Rotations

$$A'_x = |A|\cos(\theta+\varphi) = \underbrace{|A|\cos\theta}_{A_x}\cos\varphi - \underbrace{|A|\sin\theta}_{A_y}\sin\varphi$$



$$A'_y = |A|\sin(\theta+\varphi) = \underbrace{|A|\cos\theta}_{A_x}\sin\varphi + \underbrace{|A|\sin\theta}_{A_y}\cos\varphi$$

$$A'_x = A_x\cos\varphi - A_y\sin\varphi$$

$$A'_y = A_x\sin\varphi + A_y\cos\varphi$$

$$A'_z = A_z$$

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S(\varphi\vec{k})} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$\Delta\varphi \ll 1$

$$\sin\Delta\varphi \cong \Delta\varphi + \ldots$$

$$\cos\Delta\varphi \cong 1 - \frac{(\Delta\varphi)^2}{2} + \ldots$$

$$S(\Delta\varphi\vec{k}) = \begin{pmatrix} 1-\frac{(\Delta\varphi)^2}{2} & -\Delta\varphi & 0 \\ \Delta\varphi & 1-\frac{(\Delta\varphi)^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S(\Delta\varphi\vec{k}) = \begin{pmatrix} 1-\frac{(\Delta\varphi)^2}{2} & -\Delta\varphi & 0 \\ \Delta\varphi & 1-\frac{(\Delta\varphi)^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S(\Delta\varphi\vec{i}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{(\Delta\varphi)^2}{2} & -\Delta\varphi \\ 0 & \Delta\varphi & 1-\frac{(\Delta\varphi)^2}{2} \end{pmatrix} \qquad S(\Delta\varphi\vec{j}) = \begin{pmatrix} 1-\frac{(\Delta\varphi)^2}{2} & 0 & \Delta\varphi \\ 0 & 1 & 0 \\ -\Delta\varphi & 0 & 1-\frac{(\Delta\varphi)^2}{2} \end{pmatrix}$$

$$S(\Delta\varphi\vec{i})S(\Delta\varphi\vec{j}) - S(\Delta\varphi\vec{j})S(\Delta\varphi\vec{i}) = \begin{pmatrix} 0 & -(\Delta\varphi)^2 & 0 \\ (\Delta\varphi)^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = S((\Delta\varphi)^2\vec{k}) - 1$$

$$\hat{R}(\Delta\varphi\vec{k}) = e^{-\frac{i\hat{J}_z\Delta\varphi}{\hbar}} \cong 1 - \frac{i}{\hbar}\hat{J}_z\Delta\varphi - \frac{1}{2}\left(\frac{\hat{J}_z\Delta\varphi}{\hbar}\right)^2 + \dots$$

$$\hat{R}(\Delta\varphi\vec{i}) = e^{-\frac{i\hat{J}_x\Delta\varphi}{\hbar}} \cong 1 - \frac{i}{\hbar}\hat{J}_x\Delta\varphi - \frac{1}{2}\left(\frac{\hat{J}_x\Delta\varphi}{\hbar}\right)^2 + \dots$$

$$\hat{R}(\Delta\varphi\vec{j}) = e^{-\frac{i\hat{J}_y\Delta\varphi}{\hbar}} \cong 1 - \frac{i}{\hbar}\hat{J}_y\Delta\varphi - \frac{1}{2}\left(\frac{\hat{J}_y\Delta\varphi}{\hbar}\right)^2 + \dots$$

$$\hat{R}(\Delta\varphi\vec{i})\hat{R}(\Delta\varphi\vec{j})-\hat{R}(\Delta\varphi\vec{j})\hat{R}(\Delta\varphi\vec{i})=\hat{R}((\Delta\varphi)^2\vec{k})-1$$

$$\left(1-\frac{i}{\hbar}\hat{J}_x\Delta\varphi-\frac{1}{2}\left(\frac{\hat{J}_x\Delta\varphi}{\hbar}\right)^2\right)\left(1-\frac{i}{\hbar}\hat{J}_y\Delta\varphi-\frac{1}{2}\left(\frac{\hat{J}_y\Delta\varphi}{\hbar}\right)^2\right)-\left(1-\frac{i}{\hbar}\hat{J}_y\Delta\varphi-\frac{1}{2}\left(\frac{\hat{J}_y\Delta\varphi}{\hbar}\right)^2\right)\left(1-\frac{i}{\hbar}\hat{J}_x\Delta\varphi-\frac{1}{2}\left(\frac{\hat{J}_x\Delta\varphi}{\hbar}\right)^2\right)=$$

$$=\left(1-\frac{i}{\hbar}\hat{J}_y\Delta\varphi-\frac{1}{2}\left(\frac{\hat{J}_y\Delta\varphi}{\hbar}\right)^2-\frac{i}{\hbar}\hat{J}_x\Delta\varphi-\frac{(\Delta\varphi)^2}{\hbar^2}\hat{J}_x\hat{J}_y-\frac{1}{2}\left(\frac{\hat{J}_x\Delta\varphi}{\hbar}\right)^2\right)-\left(1-\frac{i}{\hbar}\hat{J}_x\Delta\varphi-\frac{1}{2}\left(\frac{\hat{J}_x\Delta\varphi}{\hbar}\right)^2-\frac{i}{\hbar}\hat{J}_y\Delta\varphi-\frac{(\Delta\varphi)^2}{\hbar^2}\hat{J}_y\hat{J}_x-\frac{1}{2}\left(\frac{\hat{J}_y\Delta\varphi}{\hbar}\right)^2\right)$$

$$=-\frac{(\Delta\varphi)^2}{\hbar^2}\hat{J}_x\hat{J}_y+\frac{(\Delta\varphi)^2}{\hbar^2}\hat{J}_y\hat{J}_x=-\frac{(\Delta\varphi)^2}{\hbar^2}\left(\hat{J}_x\hat{J}_y-\hat{J}_y\hat{J}_x\right)$$

$$\hat{R}((\Delta\varphi)^2\vec{k})-1\cong 1-\frac{i}{\hbar}\hat{J}_z(\Delta\varphi)^2-\frac{1}{2}\left(\frac{\hat{J}_z(\Delta\varphi)^2}{\hbar}\right)^2-1=-\frac{i}{\hbar}(\Delta\varphi)^2\hat{J}_z$$

$$-\frac{(\Delta\varphi)^2}{\hbar^2}\left(\hat{J}_x\hat{J}_y-\hat{J}_y\hat{J}_x\right)=-\frac{i}{\hbar}(\Delta\varphi)^2\hat{J}_z$$

$$\hat{J}_x\hat{J}_y-\hat{J}_y\hat{J}_x=i\hbar\hat{J}_z$$

$$\hat{A}\hat{B}-\hat{B}\hat{A}=[\hat{A},\hat{B}]$$

$$[\hat{B},\hat{A}]=-[\hat{A},\hat{B}]$$

$$[\hat{A},\hat{B}+\hat{C}]=\hat{A}(\hat{B}+\hat{C})-(\hat{B}+\hat{C})\hat{A}=$$

$$=\hat{A}\hat{B}+\hat{A}\hat{C}-\hat{B}\hat{A}-\hat{C}\hat{A}=[\hat{A},\hat{B}]+[\hat{A},\hat{C}]$$

$$[\hat{A},\hat{B}\hat{C}]=\hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C}+\hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A}$$

$$=(\hat{A}\hat{B}-\hat{B}\hat{A})\hat{C}+\hat{B}(\hat{A}\hat{C}-\hat{C}\hat{A})$$

$$=\hat{B}[\hat{A},\hat{C}]+[\hat{A},\hat{B}]\hat{C}$$

$$\hat{J}_x\hat{J}_y-\hat{J}_y\hat{J}_x=i\hbar\hat{J}_z$$

$$[\hat{J}_x,\hat{J}_y]=i\hbar\hat{J}_z$$

$$[\hat{J}_z,\hat{J}_x]=i\hbar\hat{J}_y$$

$$[\hat{J}_y,\hat{J}_z]=i\hbar\hat{J}_x$$

$$[\hat{J}_x,\hat{J}_z]=-i\hbar\hat{J}_y$$

$$[\hat{J}_z,\hat{J}_y]=-i\hbar\hat{J}_x$$

$$[\hat{B},\hat{A}] = -[\hat{A},\hat{B}]$$

$$[\hat{A},\hat{B}+\hat{C}] = [\hat{A},\hat{B}] + [\hat{A},\hat{C}]$$

$$[\hat{A},\hat{B}\hat{C}] = \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C}$$

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

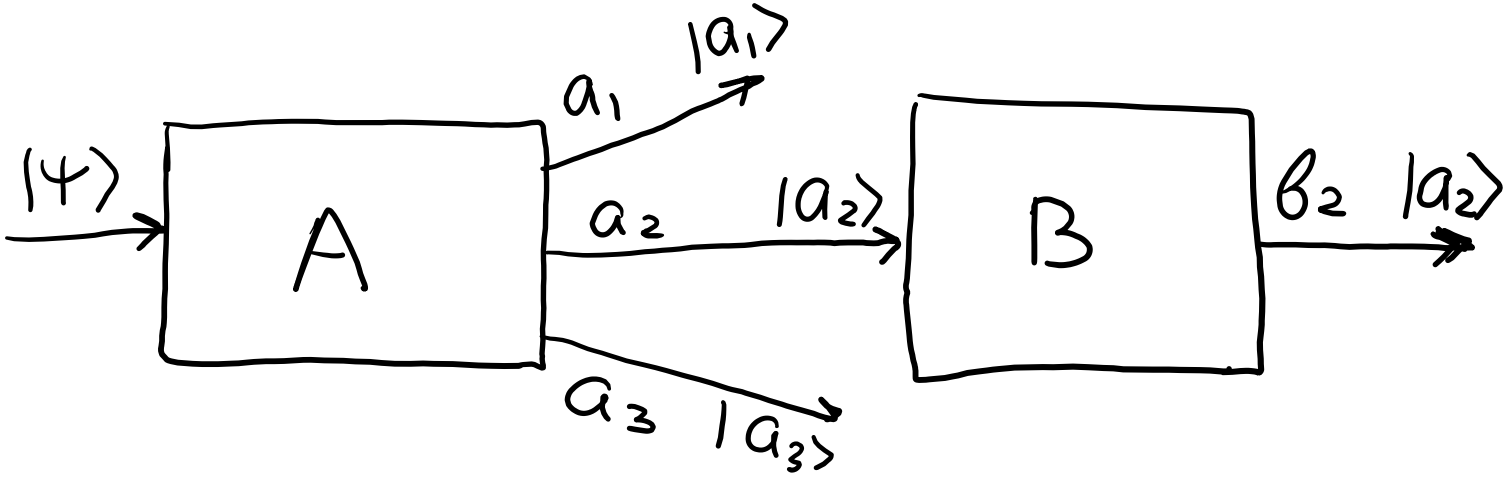
Commuting Hermitian Operators

$$[\hat{A}, \hat{B}] = 0 \Rightarrow \hat{A}\hat{B} - \hat{B}\hat{A} = 0 \Rightarrow \hat{A}\hat{B} = \hat{B}\hat{A}$$

$$\hat{A}|a\rangle = a|a\rangle$$

$$\hat{A}\left(\hat{B}|a\rangle\right) = \hat{B}\hat{A}|a\rangle = \hat{B}\,a|a\rangle = a\left(\hat{B}|a\rangle\right)$$

$$\hat{B}|a\rangle = b|a\rangle$$



$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$$

$$[\hat{J}_z, \hat{J}^2] = [\hat{J}_z, \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2] = [\hat{J}_z, \hat{J}_x^2] + [\hat{J}_z, \hat{J}_y^2] + [\hat{J}_z, \hat{J}_z^2]$$

$$[\hat{J}_z, \hat{J}_x\hat{J}_x] = \hat{J}_x\underbrace{[\hat{J}_z, \hat{J}_x]}_{i\hbar\hat{J}_y} + \underbrace{[\hat{J}_z, \hat{J}_x]}_{i\hbar\hat{J}_y}\hat{J}_x = i\hbar\hat{J}_x\hat{J}_y + i\hbar\hat{J}_y\hat{J}_x$$

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$$

$$[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$$

$$[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$$

$$[\hat{J}_z, \hat{J}_y\hat{J}_y] = \hat{J}_y\underbrace{[\hat{J}_z, \hat{J}_y]}_{-i\hbar\hat{J}_x} + \underbrace{[\hat{J}_z, \hat{J}_y]}_{-i\hbar\hat{J}_x}\hat{J}_y = -i\hbar\hat{J}_y\hat{J}_x - i\hbar\hat{J}_x\hat{J}_y$$

$$[\hat{J}_z, \hat{J}_z^2] = 0$$

$$[\hat{J}_z, \hat{J}^2] = i\hbar\hat{J}_x\hat{J}_y + i\hbar\hat{J}_y\hat{J}_x - i\hbar\hat{J}_y\hat{J}_x - i\hbar\hat{J}_x\hat{J}_y + 0 = 0$$