$\hat{A}\hat{B}|\psi\rangle = |\varphi\rangle$

$\langle\psi|\hat{B}^{+}\hat{A}^{+} = \langle\varphi|$

$(\hat{A}\hat{B})^{+} = \hat{B}^{+}\hat{A}^{+}$

$|\psi\rangle = |+z\rangle\langle+z|\psi\rangle + |-z\rangle\langle-z|\psi\rangle$

$\langle S_z\rangle = \frac{\hbar}{2}|\langle+z|\psi\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle-z|\psi\rangle|^2$

$$\underbrace{\left(\langle\psi|+z\rangle, \langle\psi|-z\rangle\right)}_{\langle\psi|}\ \underbrace{\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{\hat{J}_z}\ \underbrace{\begin{pmatrix}\langle+z|\psi\rangle\\ \langle-z|\psi\rangle\end{pmatrix}}_{|\psi\rangle} =$$

$$= \frac{\hbar}{2}\left(\langle\psi|+z\rangle, \langle\psi|-z\rangle\right)\begin{pmatrix}\langle+z|\psi\rangle\\ -\langle-z|\psi\rangle\end{pmatrix} = \frac{\hbar}{2}\left(\langle\psi|+z\rangle\langle+z|\psi\rangle - \langle\psi|-z\rangle\langle-z|\psi\rangle\right) =$$

$$= \frac{\hbar}{2}|\langle+z|\psi\rangle|^2 - \frac{\hbar}{2}|\langle-z|\psi\rangle|^2 \quad = \langle S_z\rangle$$

$\langle S_z\rangle = \langle\psi|\hat{J}_z|\psi\rangle$

$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$

p # 1.3, 1.6, 1.10, 1.14

Quantum Mechanics I (PHYS 3143)

Sample Test 1 **Name:**

Problem 1 (40 points)

A spin- ½ particle is prepared in the state vector $|\psi\rangle$. For this state vector quantum mechanics predicts the expectation values $\langle S_z\rangle = \frac{7}{50}\hbar$ and $\langle S_x\rangle = -\frac{12}{25}\hbar$.

1) **(10 points)** Circle all statements that are **TRUE**. No calculations are required for this part.

   a) ~~If, using an SGz analyzer, we experimentally measure S_z for a particle prepared in the state $|\psi\rangle$ we will obtain the value $\frac{7}{50}\hbar$ with probability 1.~~

   b) ~~If we take a single particle prepared in the state $|\psi\rangle$ and conduct 100 consecutive measurements of S_z for this particle, we will record a list with the values $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, and the experimental average value will be close to $\frac{7}{50}\hbar$.~~

   c) ~~If we take 100 identical particles – all prepared in the state $|\psi\rangle$, and conduct measurements of S_z for each particle, we will record a list with the values $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, and the experimental average value will be **exactly equal** to $\frac{7}{50}\hbar$.~~

   d) If we take 100 identical particles – all prepared in the state $|\psi\rangle$, and conduct measurements of S_z for each particle, we will record a list with the values $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, and the experimental average value will be **close** to $\frac{7}{50}\hbar$.

   e) ~~Suppose that we take 100 identical particles – all prepared in the state $|\psi\rangle$, and send each particle through a SGz analyzer followed by a SGx analyzer. For the particles exiting the SGx analyzer we will obtain an experimental average value for S_z close to $\frac{7}{50}\hbar$ and an experimental average value for S_x close to $-\frac{12}{25}\hbar$.~~ O

   f) If we take 200 identical particles – all prepared in the state $|\psi\rangle$, and conduct measurements of S_z for 100 of them, and measurements of S_x for the remaining 100 particles, we will obtain an experimental average value of S_z close to $\frac{7}{50}\hbar$ for the first group and an experimental average value of S_x close to $-\frac{12}{25}\hbar$, for the latter group.

$\langle S_z\rangle = \frac{7}{50}\hbar \qquad \langle S_x\rangle = -\frac{12}{25}\hbar$

$|\psi\rangle = C_+|+z\rangle + C_-|-z\rangle$

2) **(30 points)** Determine the state of the particle $|\psi\rangle$ as completely as possible from given information.

$\langle S_z\rangle = \frac{\hbar}{2}|C_+|^2 + \left(-\frac{\hbar}{2}\right)|C_-|^2 = \frac{7}{50}\hbar$

$|C_+|^2 - |C_-|^2 = \frac{7}{25}$ $\quad\rightarrow |C_-|^2 = 1 - \frac{16}{25} = \frac{9}{25}$

Normalization $\rightarrow$ $+\ |C_+|^2 + |C_-|^2 = 1$

$2|C_+|^2 = 1 + \frac{7}{25} = \frac{32}{25} \quad \Rightarrow \quad |C_+|^2 = \frac{16}{25}$

$|C_+| = \frac{4}{5} \qquad |C_-| = \frac{3}{5}$

$|\psi\rangle = \frac{4}{5}|+z\rangle + e^{i\theta}\cdot\frac{3}{5}|-z\rangle$

$\langle +x|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| + \frac{1}{\sqrt{2}}\langle -z|\right)\left(\frac{4}{5}|+z\rangle + e^{i\theta}\frac{3}{5}|-z\rangle\right) =$

$= \frac{4}{5\sqrt{2}} + \frac{3e^{i\theta}}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}(4+3e^{i\theta})$

$|\langle +x|\psi\rangle|^2 = \frac{1}{5\sqrt{2}}(4+3e^{i\theta})\frac{1}{5\sqrt{2}}(4+3e^{-i\theta}) = \frac{1}{25\cdot 2}(16+12e^{-i\theta}+12e^{i\theta}+9) =$

$= \frac{1}{50}\left(25 + 2\cdot 12\left(\underbrace{\frac{e^{i\theta}+e^{-i\theta}}{2}}_{=\cos\theta}\right)\right) = \frac{1}{50}(25+24\cos\theta) = \frac{1}{2} + \frac{12}{25}\cos\theta$

$|\langle -x|\psi\rangle|^2 = 1 - |\langle +x|\psi\rangle|^2 = 1 - \left(\frac{1}{2} + \frac{12}{25}\cos\theta\right) = \frac{1}{2} - \frac{12}{25}\cos\theta$

$\langle S_x\rangle = \frac{\hbar}{2}|\langle +x|\psi\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle -x|\psi\rangle|^2 = \frac{\hbar}{2}\left(\cancel{\frac{1}{2}} + \frac{12}{25}\cos\theta\right) - \frac{\hbar}{2}\left(\cancel{\frac{1}{2}} - \frac{12}{25}\cos\theta\right) =$

$= \hbar\cdot\frac{12}{25}\cos\theta = -\frac{12}{25}\hbar \leftarrow$ given

$\cos\theta = -1$

$\theta = \pi$

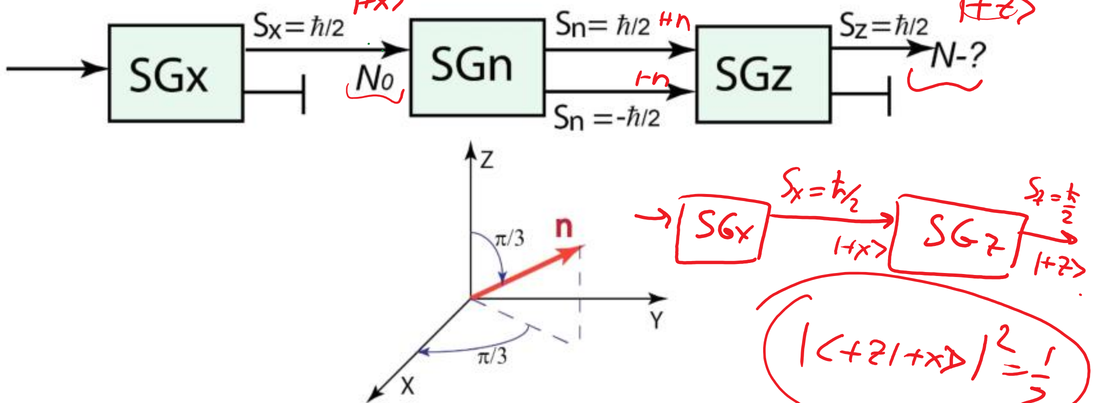
$e^{i\pi} = -1$

$|\psi\rangle = \frac{4}{5}|+z\rangle + e^{i\pi}\cdot\frac{3}{5}|-z\rangle$

$\boxed{|\psi\rangle = \frac{4}{5}|+z\rangle - \frac{3}{5}|-z\rangle}$

does not make a measurement

Problem 1 (30 points).



$$|\langle +z|+x\rangle|^2 = \frac{1}{2}$$

Figure 1

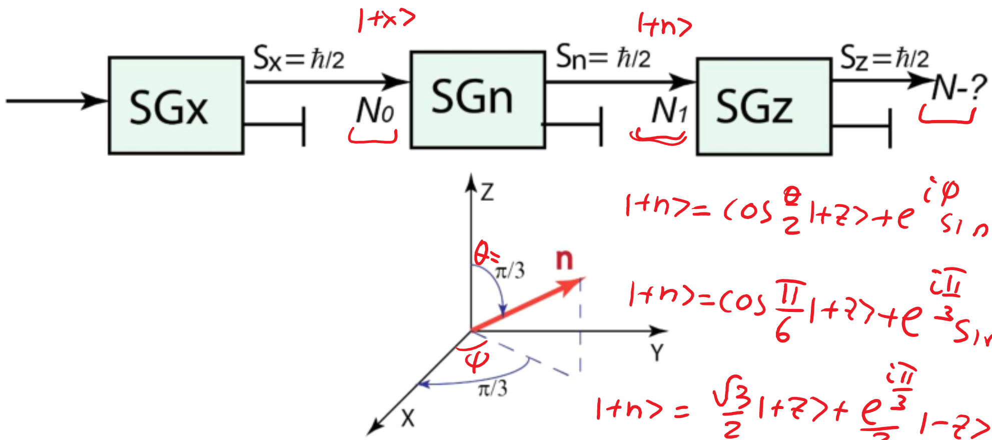
A beam of spin-½ particles is sent through a series of three Stern-Gerlach measuring devices, as illustrated in Figure 1. The first device, an SG**x** device, transmits particles with $S_x = \hbar/2$ and filters out particles with $S_x = -\hbar/2$. The second device, an SG**n** device, transmits particles with both $S_n = \hbar/2$ and $S_n = -\hbar/2$, where the direction **n** is shown in the Figure. Finally, all particles transmitted through the SG**n** device are sent through the SG**z** device which transmits particles with $S_z = \hbar/2$ and filters out particles with $S_z = -\hbar/2$.

1) **(5 points)** Does the probability for the particle transmitted by the SG**x** device to pass the SG**z** device depend on the direction of **n**? Explain.

NO

2) **(5 points)** What fraction of the particles transmitted by the first SG**x** device will survive the measurement of S_z (i.e what is N/N_0)?

$$|\langle +z|+x\rangle|^2 = \frac{1}{2}$$



$$|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle$$

$$|+n\rangle = \cos\frac{\pi}{6}|+z\rangle + e^{i\frac{\pi}{3}}\sin\frac{\pi}{6}|-z\rangle$$

$$|+n\rangle = \frac{\sqrt{3}}{2}|+z\rangle + \frac{e^{i\frac{\pi}{3}}}{2}|-z\rangle$$

Figure 2

3) **(20 Points)** The experiment is modified. Particles exiting SG**n** device with $S_n = -\hbar/2$ are discarded (see Figure 2). Only particles with $S_n = \hbar/2$ are sent through the SG**z** device. We need again to calculate the fraction of the particles transmitted by the first SG**x** device that will survive the measurement of S_z (i.e N/N_0). To do that

a) Calculate the fraction of the particles transmitted by the first SG**x** device that will survive the measurement of S_n (i.e N_1/N_0)

$$\frac{N_1}{N_0} = |\langle +n|+x\rangle|^2 - ?$$

$$\langle +n|+x\rangle = \underbrace{\left(\frac{\sqrt{3}}{2}\langle +z| + \frac{e^{-i\pi/3}}{2}\langle -z|\right)}_{\langle +n|}\underbrace{\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right)}_{|+x\rangle} =$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{e^{-i\pi/3}}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}\left(\sqrt{3} + e^{-i\frac{\pi}{3}}\right)$$

$$\frac{N_1}{N_0} = |\langle +n|+x\rangle|^2 = \frac{1}{2\sqrt{2}}\left(\sqrt{3} + e^{-i\frac{\pi}{3}}\right)\cdot\frac{1}{2\sqrt{2}}\left(\sqrt{3} + e^{i\frac{\pi}{3}}\right) = \frac{1}{8}\left(3 + \sqrt{3}e^{i\frac{\pi}{3}} + \sqrt{3}e^{-i\frac{\pi}{3}} + 1\right)$$

$$= \frac{1}{8}\left(4 + 2\cdot\sqrt{3}\left(\frac{e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}}{2}\right)\right) = \frac{1}{8}\left(4 + 2\sqrt{3}\cos\frac{\pi}{3}\right) = \boxed{\frac{1}{8}\left(4 + \sqrt{3}\right)}$$

b) Calculate the fraction of the particles transmitted by the SG**n** device that will survive the measurement of $S_{\mathbf{z}}$ (i.e N/N_1)

$$\frac{N}{N_1} = |\langle +z | +n \rangle|^2$$

$$\langle +z | +n \rangle = \langle +z | \left(\frac{\sqrt{3}}{2} |+z\rangle + \frac{e^{i\pi/3}}{2} |-z\rangle \right) = \frac{\sqrt{3}}{2}$$

$$\frac{N}{N_1} = \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{3}{4}$$

c) Multiply your answers in a) and b)

$$\frac{N}{N_0} = \frac{N}{N_1} \cdot \frac{N_1}{N_0} = \frac{3}{4} \cdot \frac{1}{8}(4 + \sqrt{3}) = \frac{12 + 3\sqrt{3}}{32} > \frac{1}{2}$$

4) **(5 Points)** (Bonus – not mandatory, may be skipped without penalty). Compare your answers in parts 3) and 1). Which one is larger? Why it is possible?

Problem 3 (40 points).

Operator $\hat{A}$ is represented in a two-dimensional basis by 2×2 matrix:

$$\hat{A} \xrightarrow[Sz\ basis]{} a\begin{pmatrix} -1 & 2i \\ -2i & 2 \end{pmatrix}$$

$$A^{+} = (A^{T})^{*} = a^{*}\begin{pmatrix} -1 & 2i \\ -2i & 2 \end{pmatrix}$$

$$A^{T} = a\begin{pmatrix} -1 & -2i \\ 2i & 2 \end{pmatrix}$$

Real

where a is some constant.

a) Is operator $\hat{A}$ Hermitian?

Yes

$$\hat{A}^{+} = a\begin{pmatrix} -1 & +2i \\ -2i & 2 \end{pmatrix}$$

Since $\hat{A}^{+} = \hat{A}$

b) Measurement of the observable A (corresponding to $\hat{A}$) is performed on the particle in the state

$$|\psi\rangle \xrightarrow[Sz\ basis]{} \sqrt{\frac{1}{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

What are possible results of such measurement?

$$\det\begin{vmatrix} -1-\lambda & 2i \\ -2i & 2-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(2-\lambda) - 2i\cdot(-2i) = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = 3, -2$$

Eigenvalues of $\hat{A}$: $3a, -2a$

Possible results of measurent: $3a, -2a$

1) Find $|3a\rangle$ and $|-2a\rangle$ in S_z Basis

2) Find $|\langle 3a|\psi\rangle|^2$ and $|\langle -2a|\psi\rangle|^2$

c) What are probabilities of each possible measurement outcome from b)?

1) Find $|3a\rangle$ $\lambda = 3$

$$\begin{pmatrix} -1-3 & 2i \\ -2i & 2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & 2i \\ -2i & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4x_1 + 2i x_2 = 0$$

$$x_1 = \frac{1}{2} i x_2$$

$$|3a\rangle \xrightarrow[S_z \text{ Basis}]{} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{c\begin{pmatrix} i \\ 2 \end{pmatrix}}$$

$$\langle 3a|3a\rangle = 1 = c^*(-i, 2)\cdot c\begin{pmatrix} i \\ 2 \end{pmatrix} = c^* \cdot c(-i\cdot i + 2\cdot 2) =$$

$$= |c|^2(1+4) = 5|c|^2 = 1$$

$$|c| = \frac{1}{\sqrt{5}}$$

$$|3a\rangle \xrightarrow[S_z \text{ Basis}]{} \underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix} i \\ 2 \end{pmatrix}}$$

$$\langle 3a|\psi\rangle = \frac{1}{\sqrt{5}}(-i, 2)\cdot\frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{50}}(-3i - 2)$$

$$|\langle 3a|\psi\rangle|^2 = \frac{1}{\sqrt{50}}(-3i-2)\frac{1}{\sqrt{50}}(+3i-2) = \frac{1}{50}(9+4) = \frac{13}{50}$$

$\frac{13}{50}$ is Prob. to get $3a$ in your measurement

$$|\langle -2a|\psi\rangle|^2 = 1 - |\langle 3a|\psi\rangle|^2 = 1 - \frac{13}{50} = \frac{37}{50}$$