$$\hat{R}(d\varphi\,\vec{k}) = 1 - \frac{i}{\hbar}\hat{J}_z\,d\varphi$$

$$\hat{J}_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$$

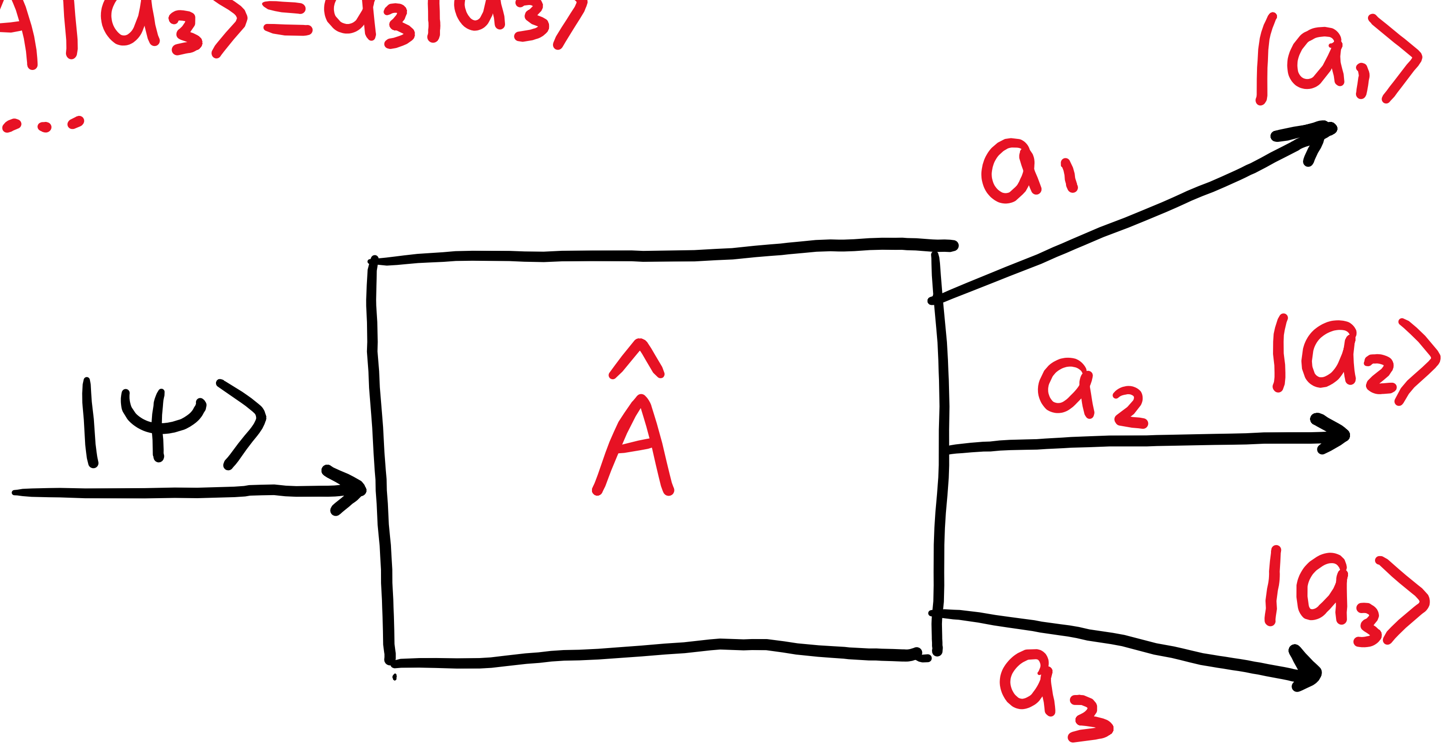
$$\hat{J}_z|-z\rangle = -\frac{\hbar}{2}|-z\rangle$$

$$\hat{A}|a_1\rangle = a_1|a_1\rangle$$

$$\hat{A}|a_2\rangle = a_2|a_2\rangle$$

$$\hat{A}|a_3\rangle = a_3|a_3\rangle$$

$\dots$



$$P = |\langle a_1|\psi\rangle|^2$$

$$P = |\langle a_2|\psi\rangle|^2$$

$$P = |\langle a_3|\psi\rangle|^2$$

$$\hat{A}|\psi\rangle = |\varphi\rangle \qquad \hat{A}\big(|+z\rangle\langle +z| + |-z\rangle\langle -z|\big)|\psi\rangle = \big(|+z\rangle\langle +z| + |-z\rangle\langle -z|\big)|\varphi\rangle$$

$$\hat{A}\big(|+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle\big) = |+z\rangle\langle +z|\varphi\rangle + |-z\rangle\langle -z|\varphi\rangle$$

$$\langle +z|\hat{A}|+z\rangle\langle +z|\psi\rangle + \langle +z|\hat{A}|-z\rangle\langle -z|\psi\rangle = \langle +z|+z\rangle\langle +z|\varphi\rangle + \langle +z|-z\rangle\langle -z|\varphi\rangle$$

$$\langle -z|\hat{A}|+z\rangle\langle +z|\psi\rangle + \langle -z|\hat{A}|-z\rangle\langle -z|\psi\rangle = \langle -z|+z\rangle\langle +z|\varphi\rangle + \langle -z|-z\rangle\langle -z|\varphi\rangle$$

$$\underbrace{\begin{pmatrix} \langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix}}_{\substack{\hat{A} \\ \text{in } S_z \text{ basis}}} \underbrace{\begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}}_{\substack{|\psi\rangle \\ \text{in } S_z \text{ basis}}} = \underbrace{\begin{pmatrix} \langle +z|\varphi\rangle \\ \langle -z|\varphi\rangle \end{pmatrix}}_{\substack{|\varphi\rangle \\ \text{in } S_z \text{ basis}}}$$

$$\hat{J}_z \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \langle +z|\hat{J}_z|+z\rangle & \langle +z|\hat{J}_z|-z\rangle \\ \langle -z|\hat{J}_z|+z\rangle & \langle -z|\hat{J}_z|-z\rangle \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{J}_z \xrightarrow[S_x \text{ basis}]{} \begin{pmatrix} \langle +x|\hat{J}_z|+x\rangle & \langle +x|\hat{J}_z|-x\rangle \\ \langle -x|\hat{J}_z|+x\rangle & \langle -x|\hat{J}_z|-x\rangle \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2}\langle +x|-x\rangle & \frac{\hbar}{2}\langle +x|+x\rangle \\ \frac{\hbar}{2}\langle -x|-x\rangle & \frac{\hbar}{2}\langle -x|+x\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{J}_z|+x\rangle = \hat{J}_z\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right) = \frac{1}{\sqrt{2}}\cdot\frac{\hbar}{2}|+z\rangle + \frac{1}{\sqrt{2}}\left(-\frac{\hbar}{2}\right)|-z\rangle = \frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle\right) = \frac{\hbar}{2}|-x\rangle$$

$$\hat{J}_z|-x\rangle = \hat{J}_z\left(\frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle\right) = \frac{1}{\sqrt{2}}\cdot\frac{\hbar}{2}|+z\rangle - \frac{1}{\sqrt{2}}\left(-\frac{\hbar}{2}\right)|-z\rangle = \frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right) = \frac{\hbar}{2}|+x\rangle$$

$$|\psi\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$$

$$|\psi\rangle \xrightarrow[S_x \text{ Basis}]{} \begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix}$$

$$|+z\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|+z\rangle \xrightarrow[S_x \text{ basis}]{} \begin{pmatrix} \langle +x|+z\rangle \\ \langle -x|+z\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\langle +x|+z\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| + \frac{1}{\sqrt{2}}\langle -z|\right)|+z\rangle = \frac{1}{\sqrt{2}}$$

$$\langle -x|+z\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| - \frac{1}{\sqrt{2}}\langle -z|\right)|+z\rangle = \frac{1}{\sqrt{2}}$$

$$\hat{J}_z \xrightarrow[S_z \text{ basis}]{} \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{J}_z \xrightarrow[S_x \text{ basis}]{} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{J}_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$$

S_z basis:

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

S_x basis:

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{R}(\phi\vec{k}) \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \langle +z|\hat{R}(\phi\vec{k})|+z\rangle & \langle +z|\hat{R}(\phi\vec{k})|-z\rangle \\ \langle -z|\hat{R}(\phi\vec{k})|+z\rangle & \langle -z|\hat{R}(\phi\vec{k})|-z\rangle \end{pmatrix}$$

$$\hat{R}(\phi\vec{k})|+z\rangle = e^{-\frac{i\phi}{2}}|+z\rangle$$

$$\hat{R}(\phi\vec{k})|-z\rangle = e^{\frac{i\phi}{2}}|-z\rangle$$

$$\hat{R}(\phi\vec{k}) \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} e^{-\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix}$$

$$\hat{R}\left(\frac{\pi}{2}\vec{k}\right)|+y\rangle = \begin{pmatrix} e^{-\frac{i\pi}{4}} & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{i\pi}{4}} \\ ie^{\frac{i\pi}{4}} \end{pmatrix} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\begin{pmatrix} 1 \\ ie^{\frac{i\pi}{2}} \end{pmatrix} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-\frac{i\pi}{4}}|-x\rangle$$

$$\langle \chi | \hat{A} | \psi \rangle = \langle \chi | \varphi \rangle \qquad \langle \chi | \varphi \rangle = \langle \varphi | \chi \rangle^*$$

$$\langle \psi | \hat{A}^\dagger | \chi \rangle = \langle \varphi | \chi \rangle \qquad \langle \psi | \hat{A}^\dagger | \chi \rangle = \langle \chi | \hat{A} | \psi \rangle^*$$

$$\hat{A} \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \langle +z | \hat{A} | +z \rangle & \langle +z | \hat{A} | -z \rangle \\ \langle -z | \hat{A} | +z \rangle & \langle -z | \hat{A} | -z \rangle \end{pmatrix}$$

$$\hat{A}^\dagger \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \langle +z | \hat{A}^\dagger | +z \rangle & \langle +z | \hat{A}^\dagger | -z \rangle \\ \langle -z | \hat{A}^\dagger | +z \rangle & \langle -z | \hat{A}^\dagger | -z \rangle \end{pmatrix} = \begin{pmatrix} \langle +z | \hat{A} | +z \rangle^* & \langle -z | \hat{A} | +z \rangle^* \\ \langle +z | \hat{A} | -z \rangle^* & \langle -z | \hat{A} | -z \rangle^* \end{pmatrix}$$

$$\hat{A}^\dagger = (\hat{A}^T)^*$$

Examples:

$$\hat{A} \xrightarrow[S_x \text{ basis}]{} \begin{pmatrix} 2 & i \\ i & 3 \end{pmatrix}$$

$$\hat{B} \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} 4 & 2 \\ 2 & i \end{pmatrix}$$

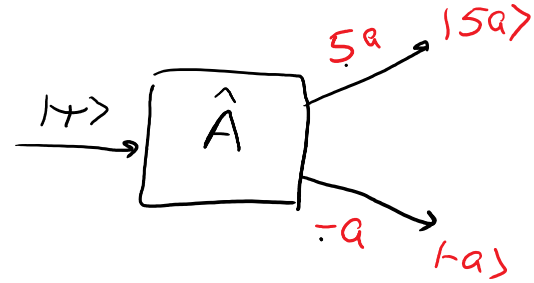
$$\hat{C} \xrightarrow[S_y \text{ basis}]{} \begin{pmatrix} 4 & 6i \\ -6i & 1 \end{pmatrix}$$

$$\begin{pmatrix} \langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|+z\rangle & \langle -x|-z\rangle \end{pmatrix}\begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle +x|+z\rangle\langle +z|\psi\rangle + \langle +x|-z\rangle\langle -z|\psi\rangle \\ \langle -x|+z\rangle\langle +z|\psi\rangle + \langle -x|-z\rangle\langle -z|\psi\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle +x|\big(|+z\rangle\langle +z| + |-z\rangle\langle -z|\big)|\psi\rangle \\ \langle -x|\big(|+z\rangle\langle +z| + |-z\rangle\langle -z|\big)|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix}$$

$$\begin{pmatrix} \langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|+z\rangle & \langle -x|-z\rangle \end{pmatrix}\begin{pmatrix} \langle +z|\hat{A}|+z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix}\begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix} = \begin{pmatrix} \langle +x|\hat{A}|+x\rangle & \langle +x|\hat{A}|-x\rangle \\ \langle -x|\hat{A}|+x\rangle & \langle -x|\hat{A}|-x\rangle \end{pmatrix}$$

Additional Example



$$\hat{A} \xrightarrow[S_z \text{ basis}]{} a\underbrace{\begin{pmatrix} 2 & 3i \\ -3i & 2 \end{pmatrix}}$$

$$|\psi\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{A}|a\rangle = \lambda|a\rangle$$

$$(\hat{A} - \lambda\hat{I})|a\rangle = 0$$

$$\det\|A - \lambda\hat{I}\| = 0$$

$$\det\begin{Vmatrix} 2-\lambda & 3i \\ -3i & 2-\lambda \end{Vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - (3i)(-3i) = 0$$

$$(2-\lambda)^2 = 3^2$$

$$2-\lambda = \pm 3$$

$$\lambda = 2 \pm 3 \qquad \lambda = 5, -1$$

Eigenvalues of $\hat{A}$

$5a, \ -1a$

$|5a\rangle$ is S_z basis

$$\begin{pmatrix} 2-5 & 3i \\ -3i & 2-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots \\ -3i & 2-5 \end{pmatrix}\begin{pmatrix} \cdots \\ x_2 \end{pmatrix} = \begin{pmatrix} \cdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3i \\ -3i & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -\cancel{3}x_1 + \cancel{3}i\,x_2 = 0$$

$$x_1 = i\,x_2$$

$$|s\,a\rangle \xrightarrow[S_? \text{ basis}]{} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \qquad \langle s\,a| = c^*(-i,\ 1)$$

$$\langle s\,a | s\,a\rangle = 1 = c^*(-i, 1)\, c \begin{pmatrix} i \\ 1 \end{pmatrix} = c^* c\, (\underbrace{-i \cdot i}_{1} + 1) = 1$$

$$|c|^2 \cdot 2 = 1 \qquad c = \frac{1}{\sqrt{2}}$$

$$|-a\rangle^{\lambda = -1} \qquad \begin{pmatrix} 2-(-1) & 3i \\ -3i & 2-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3i \\ -3i & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \cancel{3}x_1 + \cancel{3}i\,x_2 = 0$$

$$x_1 = -i\,x_2$$

$$|-a\rangle \xrightarrow[S_? \text{ basis}]{} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\langle -a | +\rangle = \frac{1}{\sqrt{2}}(1, -i) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$|\langle -a | +\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$