$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle \qquad |-x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle \qquad |-y\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle$$

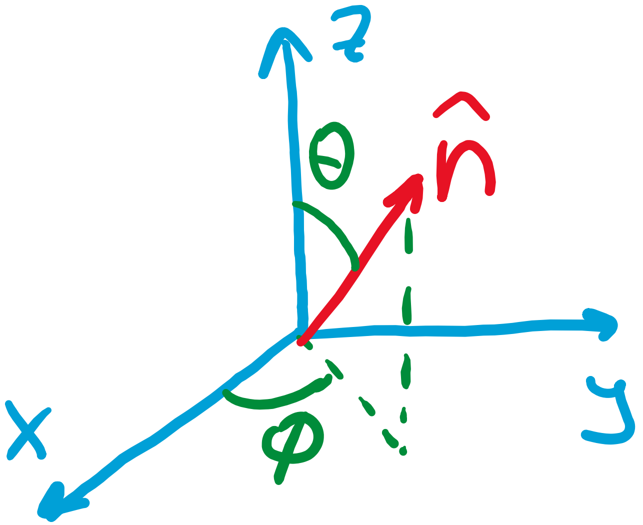
Example $|\psi\rangle = \frac{3}{5}|+z\rangle + \frac{4i}{5}|-z\rangle$ $\langle S_x\rangle$ - ?

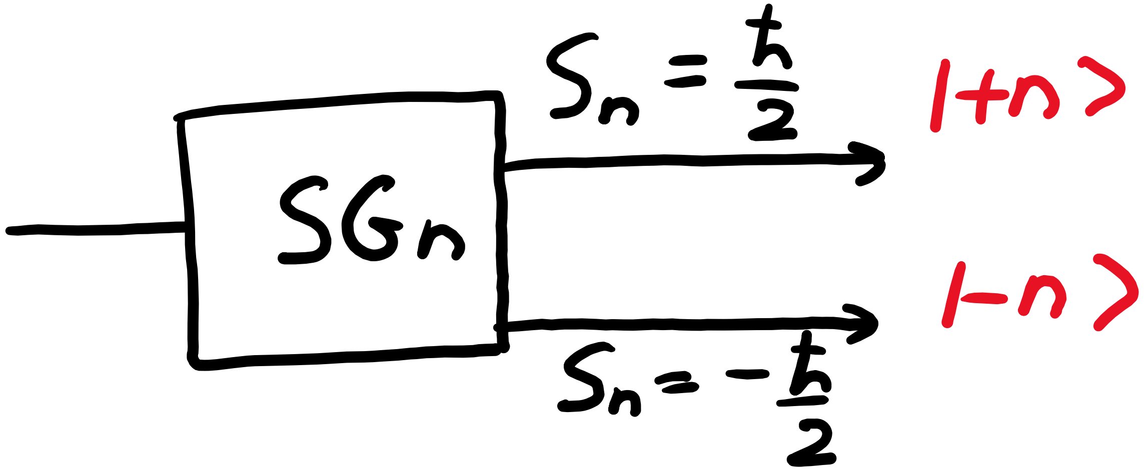
$$\langle +x|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| + \frac{1}{\sqrt{2}}\langle -z|\right)\left(\frac{3}{5}|+z\rangle + \frac{4i}{5}|-z\rangle\right) = \frac{3}{5\sqrt{2}} + \frac{4i}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}(3+4i)$$

$$|\langle +x|\psi\rangle|^2 = \frac{1}{5\sqrt{2}}(3+4i)\cdot\frac{1}{5\sqrt{2}}(3-4i) = \frac{1}{25\cdot 2}(9+16) = \frac{1}{2}$$

$$|\langle -x|\psi\rangle|^2 = 1 - |\langle +x|\psi\rangle|^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\langle S_x\rangle = \frac{\hbar}{2}|\langle +x|\psi\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle -x|\psi\rangle|^2 = \frac{\hbar}{2}\cdot\frac{1}{2} - \frac{\hbar}{2}\cdot\frac{1}{2} = 0$$





$$|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-z\rangle$$

$$|-n\rangle = \sin\frac{\theta}{2}|+z\rangle - e^{i\phi}\cos\frac{\theta}{2}|-z\rangle$$

Example: $\langle S_z\rangle = \frac{\hbar}{6}$ $\langle S_x\rangle = -\hbar\frac{\sqrt{2}}{3}$ $|\psi\rangle$ − ?

$|\psi\rangle = C_+|+z\rangle + C_-|-z\rangle$

$$\langle S_z\rangle = \frac{\hbar}{2}|C_+|^2 + \left(-\frac{\hbar}{2}\right)|C_-|^2 = \frac{\hbar}{6} \Rightarrow \left.\begin{matrix}|C_+|^2 - |C_-|^2 = \frac{1}{3}\\ |C_+|^2 + |C_-|^2 = 1\end{matrix}\right\} \; 2|C_+|^2 = \frac{4}{3}$$

$$\begin{matrix}|C_+|^2 = \frac{2}{3}\\ |C_-|^2 = \frac{1}{3}\end{matrix} \Rightarrow \begin{matrix}C_+ = \sqrt{\frac{2}{3}}\\ C_- = \sqrt{\frac{1}{3}}\,e^{i\theta}\end{matrix} \qquad |\psi\rangle = \sqrt{\frac{2}{3}}|+z\rangle + \sqrt{\frac{1}{3}}e^{i\theta}|-z\rangle$$

$$\langle +x|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| + \frac{1}{\sqrt{2}}\langle -z|\right)\left(\sqrt{\frac{2}{3}}|+z\rangle + \sqrt{\frac{1}{3}}e^{i\theta}|-z\rangle\right) = \sqrt{\frac{2}{6}} + \sqrt{\frac{1}{6}}e^{i\theta} = \frac{1}{\sqrt{6}}\left(\sqrt{2} + e^{i\theta}\right)$$

$$|\langle +x|\psi\rangle|^2 = \frac{1}{\sqrt{6}}\left(\sqrt{2}+e^{i\theta}\right)\cdot\frac{1}{\sqrt{6}}\left(\sqrt{2}+e^{-i\theta}\right) = \frac{1}{6}\left(2+\sqrt{2}e^{-i\theta}+\sqrt{2}e^{i\theta}+1\right) = \frac{1}{6}\left(3+2\sqrt{2}\,\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = \frac{1}{2}+\frac{\sqrt{2}}{3}\cos\theta$$

$$|\langle -x|\psi\rangle|^2 = 1 - |\langle +x|\psi\rangle|^2 = 1 - \left(\frac{1}{2}+\frac{\sqrt{2}}{3}\cos\theta\right) = \frac{1}{2} - \frac{\sqrt{2}}{3}\cos\theta$$

$$|\psi\rangle = \sqrt{\frac{2}{3}}|+z\rangle - \sqrt{\frac{1}{3}}|-z\rangle$$

$$\langle S_x\rangle = \frac{\hbar}{2}|\langle +x|\psi\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle -x|\psi\rangle|^2 = \frac{\hbar}{2}\left(\frac{1}{2}+\frac{\sqrt{2}}{3}\cos\theta\right) - \frac{\hbar}{2}\left(\frac{1}{2}-\frac{\sqrt{2}}{3}\cos\theta\right) = \hbar\frac{\sqrt{2}}{3}\cos\theta = -\hbar\frac{\sqrt{2}}{3} \Rightarrow \theta = \pi$$

$$|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle = |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle$$

$$|\psi\rangle \xrightarrow[S_z \text{ Basis}]{} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$$

Examples:

$$|+z\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-z\rangle \xrightarrow[S_z \text{ Basis}]{} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |+x\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad |-x\rangle \xrightarrow[S_z \text{ basis}]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\langle\psi| = c_+^*\langle +z| + c_-^*\langle -z| = \langle\psi|+z\rangle\langle +z| + \langle\psi|-z\rangle\langle -z|$$

$$\langle\psi| \xrightarrow[S_z \text{ Basis}]{} (c_+^*, c_-^*) = (\langle\psi|+z\rangle, \langle\psi|-z\rangle)$$

Examples:

$$\langle +y| \xrightarrow[S_z \text{ Basis}]{} \left(\frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right) \qquad \langle -y| \xrightarrow[S_z \text{ Basis}]{} \frac{1}{\sqrt{2}}(1, i)$$

$$\langle +y|+y\rangle = \frac{1}{\sqrt{2}}(1,-i)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}(1+1) = 1$$

$$\langle -y|+y\rangle = \frac{1}{\sqrt{2}}(1,i)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}(1-1) = 0$$

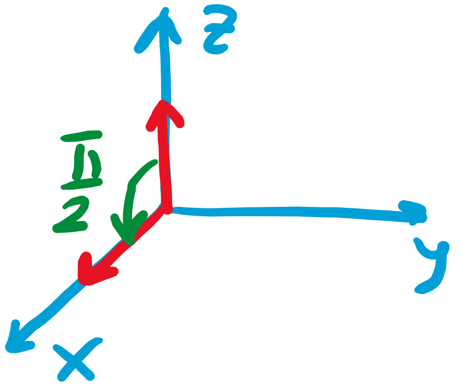
$$\langle -x|-y\rangle = \frac{1}{\sqrt{2}}(1,-1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{2}(1+i)$$

$$|\psi\rangle = \sqrt{\tfrac{2}{3}}|+z\rangle - \sqrt{\tfrac{1}{3}}|-z\rangle$$

$$\langle +x|\psi\rangle = \frac{1}{\sqrt{2}}(1,1)\frac{1}{\sqrt{3}}\begin{pmatrix}\sqrt{2}\\ -1\end{pmatrix} = \frac{1}{\sqrt{6}}(\sqrt{2}-1)$$

Functions in Algebra: $f(x) = y$

Operators $\hat{A}|\psi\rangle = |\varphi\rangle$



Rotation Operator $|+x\rangle = \hat{R}\left(\frac{\pi}{2}\vec{j}\right)|+z\rangle$

$$\langle +x| \stackrel{?}{=} \langle +z|\hat{R}\left(\frac{\pi}{2}\vec{j}\right)$$

$$\langle +x|+x\rangle = \langle +z|\underbrace{\hat{R}\left(\frac{\pi}{2}\vec{j}\right)\hat{R}\left(\frac{\pi}{2}\vec{j}\right)}_{\hat{R}(\pi\vec{j})}|+z\rangle = \langle +z|\hat{R}(\pi\vec{j})|+z\rangle = \langle +z|-z\rangle = 0$$

$$\langle +x| = \langle +z|\hat{R}^{\dagger}\left(\frac{\pi}{2}\vec{j}\right)$$

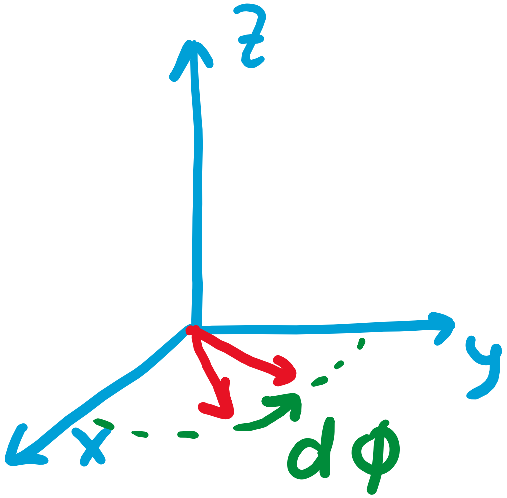
$$\langle +x|+x\rangle = \langle +z|\underbrace{\hat{R}^{\dagger}\left(\frac{\pi}{2}\vec{j}\right)\hat{R}\left(\frac{\pi}{2}\vec{j}\right)}_{\hat{1}}|+z\rangle = \langle +z|+z\rangle$$

$\hat{1}|\psi\rangle = |\psi\rangle$

Unitary: $\hat{A}^{\dagger}\hat{A} = \hat{1}$

$$\hat{A}|\psi\rangle = |\varphi\rangle \Rightarrow \langle\psi|\hat{A}^{+} = \langle\varphi|$$

$$\hat{R}(d\phi\vec{k}) = 1 - \frac{i}{\hbar}\hat{J}_z\, d\phi$$



$$\hat{R}^{+}(d\phi\vec{k}) = 1 + \frac{i}{\hbar}\hat{J}_z^{+}\, d\phi$$

$$\hat{R}(d\phi\vec{k})\,\hat{R}^{+}(d\phi k) = \left(1 - \frac{i}{\hbar}\hat{J}_z\, d\phi\right)\left(1 + \frac{i}{\hbar}\hat{J}_z^{+}\, d\phi\right)$$

$$= 1 + \frac{i}{\hbar}\hat{J}_z^{+}\, d\phi - \frac{i}{\hbar}\hat{J}_z\, d\phi + \frac{d\phi^2}{\hbar^2}\hat{J}_z\hat{J}_z^{+} = 1 + \frac{i\, d\phi}{\hbar}\left(\hat{J}_z^{+} - \hat{J}_z\right) + \frac{d\phi^2}{\hbar^2}\hat{J}_z\hat{J}_z^{+}$$

$$\hat{J}_z^{+} = \hat{J}_z$$

Hermitian Operator: $\hat{A}^{+} = \hat{A}$