$$|\psi\rangle = c_+ |+z\rangle + c_- |-z\rangle$$

Ket vector

State vector

Probability amplitudes

Probability to measure $S_z = \frac{\hbar}{2}$: $|c_+|^2$

Probability to measure $S_z = -\frac{\hbar}{2}$: $|c_-|^2$

Hilbert vector space

How do we multiply vectors?

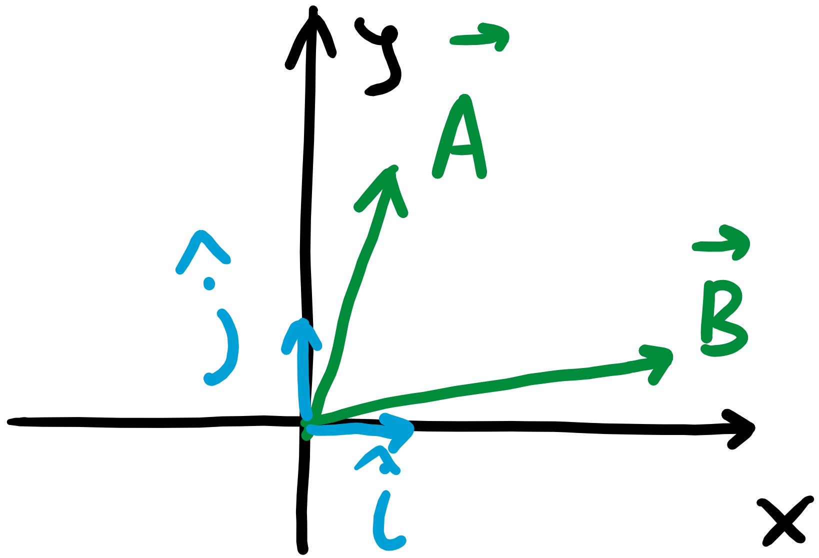
For every ket $|\psi\rangle$ there is a bra vector $\langle\psi|$

$\langle +z|+z\rangle = 1$

$\langle -z|+z\rangle = 0$

$\langle +z|-z\rangle = 0$

$\langle -z|-z\rangle = 1$



$\hat{i}\cdot\hat{i} = 1$

$\hat{i}\cdot\hat{j} = 0$

$\hat{j}\cdot\hat{j} = 1$

$\vec{A} = A_x\hat{i} + A_y\hat{j}$

$\vec{B} = B_x\hat{i} + B_y\hat{j}$

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j})(B_x\hat{i} + B_y\hat{j}) = A_xB_x + A_yB_y$$

$$|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle$$

$$\langle +z|\psi\rangle = c_+\langle +z|+z\rangle + c_-\langle +z|-z\rangle = c_+$$

$$\langle -z|\psi\rangle = c_+\langle -z|+z\rangle + c_-\langle -z|-z\rangle = c_-$$

$$|\psi\rangle = \langle +z|\psi\rangle|+z\rangle + \langle -z|\psi\rangle|-z\rangle = |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle$$

$$\langle\psi| = c_+'\langle +z| + c_-'\langle -z|$$

$$1 = \langle\psi|\psi\rangle = \left(c_+'\langle +z| + c_-'\langle -z|\right)\left(c_+|+z\rangle + c_-|-z\rangle\right)$$

$$= c_+'c_+\langle +z|+z\rangle + c_+'c_-\langle +z|-z\rangle + c_-'c_+\langle -z|+z\rangle + c_-'c_-\langle -z|-z\rangle$$

$$= c_+'c_+ + c_-'c_- = 1 \qquad \Rightarrow c_+' = c_+^* \qquad c_-' = c_-^*$$

$$\langle\psi|+z\rangle = \langle +z|\psi\rangle^*$$

$$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$$

$$\langle\psi|\psi\rangle = c_+ c_+^* + c_- c_-^* = |c_+|^2 + |c_-|^2 = 1$$

Example:

$$\langle\psi| = \frac{3}{5}\langle +z| - \frac{4i}{5}\langle -z|$$

$$|\psi\rangle = \frac{3}{5}|+z\rangle + \frac{4i}{5}|-z\rangle$$

$$\langle +z|\psi\rangle = \frac{3}{5} \qquad |\langle +z|\psi\rangle|^2 = \frac{9}{25}$$

$$\langle -z|\psi\rangle = \frac{4i}{5} \qquad |\langle -z|\psi\rangle|^2 = \frac{4i}{5}\cdot\frac{4(-i)}{5} = \frac{16}{25}$$

Expectation value

$$\langle S_z \rangle = \frac{\hbar}{2} \cdot c_+ c_+^* + \left(-\frac{\hbar}{2}\right) \cdot c_- c_-^*$$

$$|\psi\rangle = \frac{3}{5}|+z\rangle + \frac{4i}{5}|-z\rangle$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cdot \frac{9}{25} + \left(-\frac{\hbar}{2}\right) \cdot \frac{16}{25} = -\frac{7\hbar}{50}$$

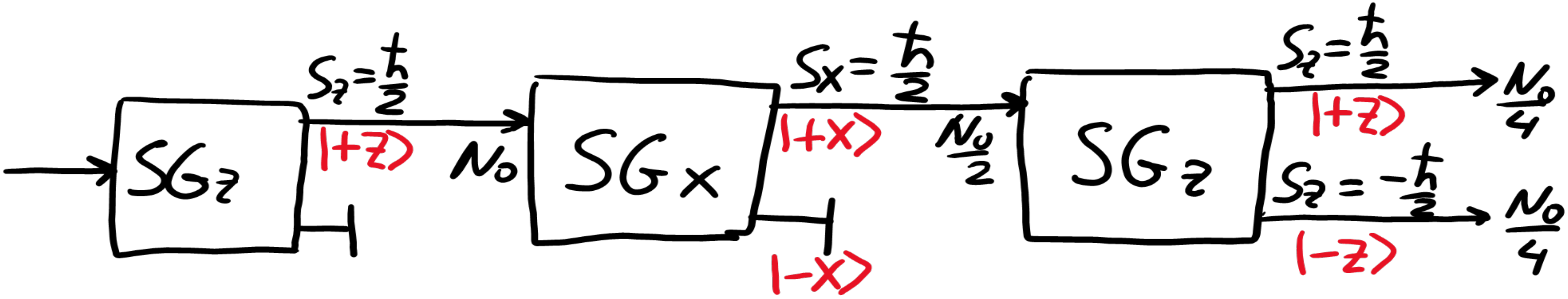
$$(\Delta S)^2 = \langle (S_z - \langle S_z \rangle)^2 \rangle = \langle S_z^2 - 2 S_z \langle S_z \rangle + \langle S_z \rangle^2 \rangle =$$

$$= \langle S_z^2 \rangle - 2\langle S_z \rangle \langle S_z \rangle + \langle S_z \rangle^2 = \langle S_z^2 \rangle - \langle S_z \rangle^2$$

$$\langle S_z^2 \rangle = \left(\frac{\hbar}{2}\right)^2 c_+ c_+^* + \left(-\frac{\hbar}{2}\right)^2 c_- c_-^* = \frac{\hbar^2}{4}\left(c_+ c_+^* + c_- c_-^*\right) = \frac{\hbar^2}{4}$$

$$(\Delta S)^2 = \frac{\hbar^2}{4} - \left(\frac{7\hbar}{50}\right)^2 = \frac{576\hbar^2}{2500}$$

$$\Delta S = \frac{12}{25}\hbar$$



$$|+x\rangle = c_+|+z\rangle + c_-|-z\rangle$$

$$|c_+|^2 = \frac{1}{2} \qquad \Rightarrow \qquad \cancel{c_+ = \frac{1}{\sqrt{2}}} \qquad\qquad c_+ = \frac{e^{i\delta_+}}{\sqrt{2}}$$

$$|c_-|^2 = \frac{1}{2} \qquad\qquad \cancel{c_- = \frac{1}{\sqrt{2}}} \qquad\qquad c_- = \frac{e^{i\delta_-}}{\sqrt{2}}$$

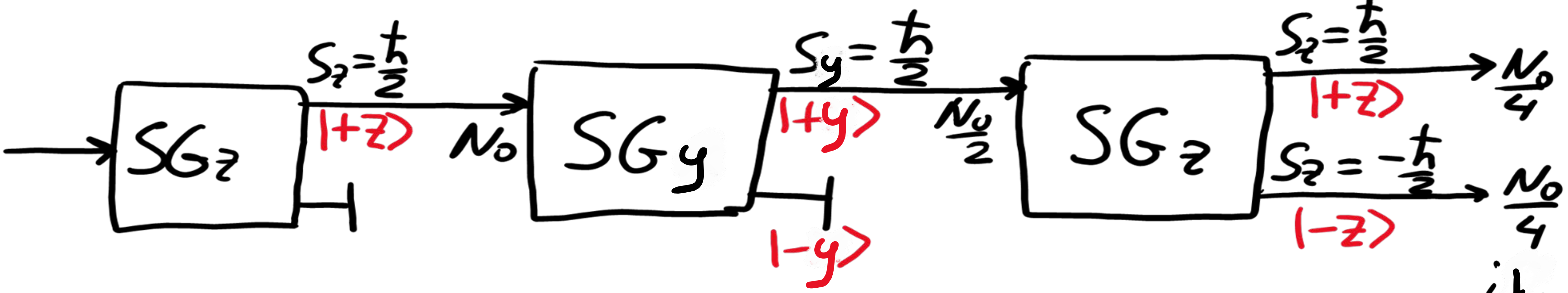
$$|c_+|^2 = c_+c_+^* = \frac{e^{i\delta_+}}{\sqrt{2}} \cdot \frac{e^{-i\delta_+}}{\sqrt{2}} = \frac{e^{0}}{2} = \frac{1}{2}$$

$$|+x\rangle = \frac{e^{i\delta_+}}{\sqrt{2}}|+z\rangle + \frac{e^{i\delta_-}}{\sqrt{2}}|-z\rangle$$

$$\delta$$

$$|+x\rangle = e^{i\delta_+}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i(\delta_- - \delta_+)}}{\sqrt{2}}|-z\rangle\right)$$

$$|+x\rangle = e^{i\delta_+}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i\delta}}{\sqrt{2}}|-z\rangle\right)$$

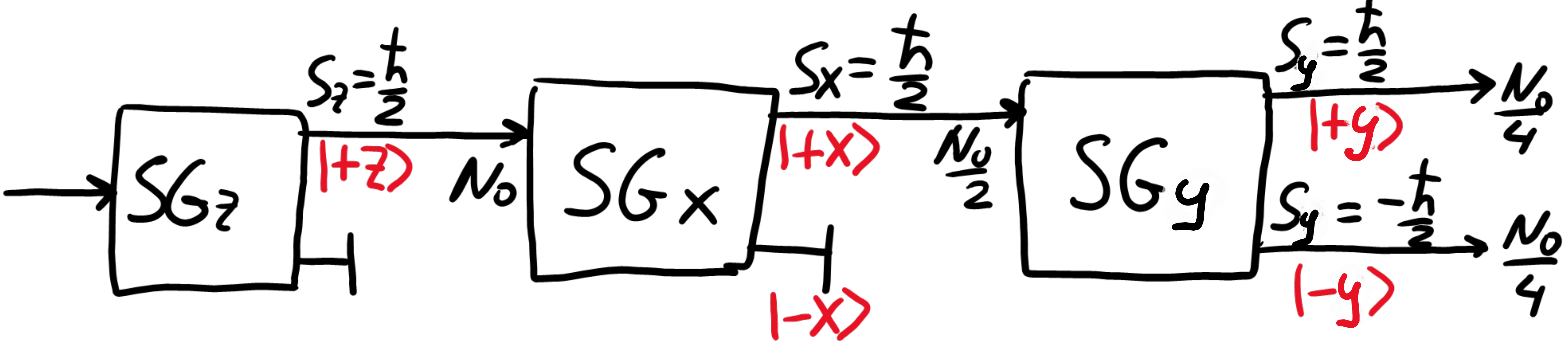


$$|+y\rangle = C_+|+z\rangle + C_-|-z\rangle$$

$$|C_+|^2 = \frac{1}{2} \qquad |C_-|^2 = \frac{1}{2} \quad \Rightarrow \quad C_+ = \frac{e^{i\gamma_+}}{\sqrt{2}} \qquad C_- = \frac{e^{i\gamma_-}}{\sqrt{2}}$$

$$|+y\rangle = e^{i\gamma_+}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i\overset{\gamma}{(\gamma_- - \gamma_+)}}}{\sqrt{2}}|-z\rangle\right)$$

$$|+y\rangle = e^{i\gamma_+}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i\gamma}}{\sqrt{2}}|-z\rangle\right)$$



$$|\langle +y|+x\rangle|^2 = \frac{1}{2}$$

$$|+y\rangle = e^{i\gamma_+}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i\gamma}}{\sqrt{2}}|-z\rangle\right)$$

$$\langle +y| = e^{-i\gamma_+}\left(\frac{1}{\sqrt{2}}\langle +z| + \frac{e^{-i\gamma}}{\sqrt{2}}\langle -z|\right)$$

$$|+x\rangle = e^{i\delta_+}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i\delta}}{\sqrt{2}}|-z\rangle\right)$$

$$\langle +y|+x\rangle = e^{-i\gamma_+}\left(\frac{1}{\sqrt{2}}\langle +z| + \frac{e^{-i\gamma}}{\sqrt{2}}\langle -z|\right) e^{i\delta_+}\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i\delta}}{\sqrt{2}}|-z\rangle\right)$$

$$= e^{i(\delta_+ - \gamma_+)}\left(\frac{1}{2} + \frac{e^{i(\delta-\gamma)}}{2}\right)$$

$$|\langle +y|+x\rangle|^2 = e^{i(\delta_+ - \gamma_+)}\left(\frac{1}{2} + \frac{e^{i(\delta-\gamma)}}{2}\right) e^{-i(\delta_+ - \gamma_+)}\left(\frac{1}{2} + \frac{e^{-i(\delta-\gamma)}}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{4}e^{-i(\delta-\gamma)} + \frac{1}{4}e^{i(\delta-\gamma)} + \frac{1}{4} = \frac{1}{2} + \frac{1}{2}\left(\frac{e^{i(\delta-\gamma)} + e^{-i(\delta-\gamma)}}{2}\right) = \frac{1}{2}\left(1+\cos(\delta-\gamma)\right) = \frac{1}{2}$$

$$\delta - \gamma = \pm\frac{\pi}{2} \qquad \delta = 0 \qquad \gamma = \frac{\pi}{2}$$ (convention)

$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{e^{i\pi/2}}{\sqrt{2}}|-z\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle$$

$$|-x\rangle = C_+|+z\rangle + C_-|-z\rangle$$

$$0 = \langle +x|-x\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| + \frac{1}{\sqrt{2}}\langle -z|\right)\left(C_+|+z\rangle + C_-|-z\rangle\right) = \frac{1}{\sqrt{2}}C_+ + \frac{1}{\sqrt{2}}C_-$$

$$C_+ = -C_- \;;\; |C_+|^2 + |C_-|^2 = 1 \Rightarrow 2|C_+|^2 = 1 \Rightarrow C_+ = \frac{e^{i\delta}}{\sqrt{2}}$$

$$|-x\rangle = e^{i\delta}\left(\frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle\right) \Rightarrow |-x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle$$

$$|-y\rangle = C_+|+z\rangle + C_-|-z\rangle$$

$$0 = \langle +y|-y\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| - \frac{i}{\sqrt{2}}\langle -z|\right)\left(C_+|+z\rangle + C_-|-z\rangle\right) = \frac{1}{\sqrt{2}}C_+ - \frac{i}{\sqrt{2}}C_-$$

$$C_+ = iC_- \quad |C_+|^2 + |C_-|^2 = 1 \Rightarrow 2|C_+|^2 = 1 \Rightarrow C_+ = \frac{e^{i\delta}}{\sqrt{2}}$$

$$C_- = -iC_+ = -\frac{ie^{i\delta}}{\sqrt{2}}$$

$$|-y\rangle = \frac{e^{i\delta}}{\sqrt{2}}|+z\rangle - \frac{ie^{i\delta}}{\sqrt{2}}|-z\rangle \Rightarrow |-y\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle$$

$$|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle \qquad |-x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle \qquad |-y\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle$$

Example $\quad |\psi\rangle = \frac{3}{5}|+z\rangle + \frac{4i}{5}|-z\rangle \quad \langle S_x\rangle - ?$

$$\langle +x|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| + \frac{1}{\sqrt{2}}\langle -z|\right)\left(\frac{3}{5}|+z\rangle + \frac{4i}{5}|-z\rangle\right) = \frac{3}{5\sqrt{2}} + \frac{4i}{5\sqrt{2}} = \frac{1}{5\sqrt{2}}(3+4i)$$

$$|\langle +x|\psi\rangle|^2 = \frac{1}{5\sqrt{2}}(3+4i)\cdot\frac{1}{5\sqrt{2}}(3-4i) = \frac{1}{25\cdot 2}(9+16) = \frac{1}{2}$$

$$|\langle -x|\psi\rangle|^2 = 1 - |\langle +x|\psi\rangle|^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\langle S_x\rangle = \frac{\hbar}{2}|\langle +x|\psi\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle -x|\psi\rangle|^2 = \frac{\hbar}{2}\cdot\frac{1}{2} - \frac{\hbar}{2}\cdot\frac{1}{2} = 0$$