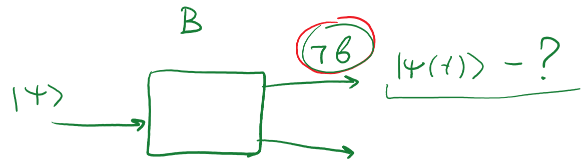


$$\hat{B} \xrightarrow[S_z\ basis]{} b\begin{pmatrix} 6 & 3i \\ -3i & -2 \end{pmatrix}$$

$|\psi(0)\rangle \rightarrow$ eigenvector of $\hat{B}$ corresponding to eigenvalue $7b$

$$\begin{pmatrix} 6-7 & 3i \\ -3i & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x_1 + 3i\,x_2 = 0$

$x_1 = 3i\,x_2$

$$|\psi(0)\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C\begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

$$1 = \langle\psi(0)|\psi(0)\rangle = C^*(-3i, 1)\, C\begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

$1 = C^*C\,(9+1)$

$1 = 10\,|C|^2 \qquad |C|^2 = \frac{1}{10}$

$|C| = \frac{1}{\sqrt{10}}$

Quantum Mechanics I (PHYS 3143A) Spring 2020 **GOOD LUCK!**

Final Exam **Name:**

You may use any printed/written materials as well as electronic versions of the textbook, your notes and class notes. Interaction/collaboration with anyone else during the test is an Honor Code violation.

Problem 1 (25 points).

A spin-1/2 particle is initially in the state

$$|\psi(0)\rangle = |+z\rangle$$

The Hamiltonian is represented by the matrix

$$\hat{H} \xrightarrow[S_z\ basis]{} E_0\begin{pmatrix} -3 & 4i \\ -4i & 3 \end{pmatrix}$$ where E_0 is a constant with the dimensions of energy.

a) Determine the state of the system $|\psi(t)\rangle$ at time t in S_z basis.

1) Find Eigenvalues of $\hat{H}$

$$\det\begin{vmatrix} -3-\lambda & 4i \\ -4i & 3-\lambda \end{vmatrix} = 0$$

$(-3-\lambda)(3-\lambda) - 4i(-4i) = 0$

$\lambda^2 - 9 - 16 = 0$

$\lambda^2 = 25$

$\lambda = \pm 5$

Eigenvalues of $\hat{H}$: $5E_0$, $-5E_0$

2) Find Eigenstates of $\hat{H}$:

a) $\lambda = 5$

$$\begin{pmatrix} -3-5 & 4i \\ -4i & 3-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$-8x_1 + 4i\,x_2 = 0 \Rightarrow x_2 = -2i\,x_1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

$$|5E_0\rangle \xrightarrow[S_z\ basis]{} \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

b) $\lambda = -5$

$$\begin{pmatrix} -3+5 & 4i \\ -4i & 3+5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$2x_1 + 4i\,x_2 = 0 \Rightarrow x_1 = -2i\,x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

$$|-5E_0\rangle \xrightarrow[S_z\ basis]{} \frac{1}{\sqrt{5}}\begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

3) Represent $|\psi(0)\rangle$ in the basis of the eigenstates of $\hat{H}$:

$$|\psi(0)\rangle = |5E_0\rangle\langle 5E_0|\psi(0)\rangle + |-5E_0\rangle\langle -5E_0|\psi(0)\rangle$$

3)

$$|\psi(0)\rangle = |5E_0\rangle\langle 5E_0|\psi(0)\rangle + |-5E_0\rangle\langle -5E_0|\psi(0)\rangle$$

$$\langle 5E_0|\psi(0)\rangle = \frac{1}{\sqrt{5}}(1, 2i)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{5}}$$

$$\langle -5E_0|\psi(0)\rangle = \frac{1}{\sqrt{5}}(2i, 1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{2i}{\sqrt{5}}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{5}}|5E_0\rangle + \frac{2i}{\sqrt{5}}|-5E_0\rangle$$

4) Time evolution.

$$|\psi(t)\rangle = \frac{1}{\sqrt{5}}e^{-\frac{i5E_0 t}{\hbar}}|5E_0\rangle + \frac{2i}{\sqrt{5}}e^{\frac{i5E_0 t}{\hbar}}|-5E_0\rangle$$

5) Switch to S_z basis

$$|\psi(t)\rangle = \frac{1}{\sqrt{5}}e^{-\frac{i5E_0 t}{\hbar}}\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2i\end{pmatrix} + \frac{2i}{\sqrt{5}}e^{\frac{i5E_0 t}{\hbar}}\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}-2i\\1\end{pmatrix} = \frac{e^{-\frac{i5E_0 t}{\hbar}}}{5}\begin{pmatrix}1\\-2i\end{pmatrix} + \frac{2ie^{\frac{i5E_0 t}{\hbar}}}{5}\begin{pmatrix}-2i\\1\end{pmatrix}$$

$$|\psi(t)\rangle = \frac{1}{5}\begin{pmatrix}e^{-\frac{i5E_0 t}{\hbar}} + 4e^{\frac{i5E_0 t}{\hbar}}\\ -2ie^{-\frac{i5E_0 t}{\hbar}} + 2ie^{\frac{i5E_0 t}{\hbar}}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}e^{-\frac{i5E_0 t}{\hbar}} + 4e^{\frac{i5E_0 t}{\hbar}}\\ (2i)(2i)\underbrace{\left(\frac{e^{\frac{i5E_0 t}{\hbar}} - e^{-\frac{i5E_0 t}{\hbar}}}{2i}\right)}_{\sin\left(\frac{5E_0 t}{\hbar}\right)}\end{pmatrix}$$

$$|\psi(t)\rangle = \frac{1}{5}\begin{pmatrix}e^{-\frac{i5E_0 t}{\hbar}} + 4e^{\frac{i5E_0 t}{\hbar}}\\ -4\sin\left(\frac{5E_0 t}{\hbar}\right)\end{pmatrix}$$

b) Find the probability to find particle at the state $|-z\rangle$ at time t.

$$\langle -z | \psi(t)\rangle = (0,1)\,\frac{1}{5}\begin{pmatrix} e^{\frac{-i5E_0t}{\hbar}} + 4e^{\frac{i5E_0t}{\hbar}} \\ -4\sin\left(\frac{5E_0t}{\hbar}\right)\end{pmatrix} = -\frac{4}{5}\sin\left(\frac{5E_0t}{\hbar}\right)$$

$$|\langle -z|\psi(t)\rangle|^2 = \frac{16}{25}\sin^2\left(\frac{5E_0t}{\hbar}\right)$$

Problem 2 (25 points)

Note: Useful Infinite Summation Identities:

$$\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

A quantum particle of mass m is in the one-dimensional potential energy well

$$V(x)=\begin{cases}0 & for\ 0<x<L\\ \infty & elsewhere\end{cases}$$

The state of the particle at time $t=0$ is

$$|\psi(0)\rangle = A\sum_{n=1}^{\infty}\frac{1}{n^3}|n\rangle = \sum_{n=1}^{\infty}\frac{A}{n^3}|n\rangle$$

probability amplitude

where $|n\rangle$ are the eigenstates of the Hamiltonian.

a) Find the normalization constant A

$$\boxed{\frac{A}{n^3} = \langle n|\psi(0)\rangle}$$

$$\langle\psi(0)|\psi(0)\rangle = 1$$

Probability : $|\langle n|\psi(0)\rangle|^2 = \frac{A^2}{n^6}$

$$\sum_{n=1}^{\infty}\frac{A^2}{n^6} = 1 \Rightarrow A^2\sum_{n=1}^{\infty}\frac{1}{n^6} = 1 \qquad A^2\cdot\frac{\pi^6}{945} = 1$$

$$A = \sqrt{\frac{945}{\pi^6}} = \boxed{\frac{3\sqrt{105}}{\pi^3}}$$

b) Write an expression for $|\psi(t)\rangle$

$$|\psi(0)\rangle = \frac{3\sqrt{105}}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}|n\rangle$$

$$|\psi(t)\rangle = \frac{3\sqrt{105}}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}e^{-\frac{iE_nt}{\hbar}}|n\rangle \qquad E_n = \frac{\hbar^2\pi^2n^2}{2mL^2}$$

c) **(Bonus 3 Points)** Calculate expectation value of the total energy $\langle E\rangle$

$$\langle E\rangle = \sum_{n=1}^{\infty}|\underbrace{\langle n|\psi(0)\rangle}_{}|^2 E_n = \sum_{n=1}^{\infty}\left(\frac{A}{n^3}\right)^2\cdot\frac{\hbar^2\pi^2n^2}{2mL^2} =$$

$e^{-\frac{iE_nt}{\hbar}}$

$$= \sum_{n=1}^{\infty}\frac{945}{\pi^6}\cdot\frac{1}{n^6}\cdot\frac{\hbar^2\pi^2n^2}{2mL^2} = \frac{945\hbar^2}{\pi^4\cdot 2mL^2}\sum_{n=1}^{\infty}\frac{1}{n^4} = \boxed{\frac{21\hbar^2}{4mL^2}}$$

$\frac{\pi^4}{90}$

Problem 3 (25 points)

A particle of mass m in the one-dimensional harmonic oscillator at time t=0 is in a state for which a measurement of the energy yields the values $\hbar\omega/2$ and $5\hbar\omega/2$ with probabilities 1/3 and 2/3 correspondingly. The uncertainty in the momentum of the particle in this state at time t=0 is $\Delta p_x = \sqrt{\frac{5m\omega\hbar}{2}}$. This information specifies the state of the particle completely.

a) (25 Points) Determine the state of the particle **at time *t*=0** and **at time *t*.**

(Hint: compute $\hat{p}_x|\psi(0)\rangle$ using $\hat{a}$ and $\hat{a}^+$ and then compute $\langle p_x^2\rangle$ and Δp_x)

$n=0$ $\quad$ $n=2$

$$E_n = \hbar\omega(n+\tfrac{1}{2})$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{i\theta}|2\rangle$$

$$\hat{p}_x|\psi(0)\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^+)\left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{i\theta}|2\rangle\right) =$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{\frac{2}{3}}e^{i\theta}\sqrt{2}\cdot|1\rangle - \frac{1}{\sqrt{3}}\cdot 1\cdot|1\rangle - \sqrt{\frac{2}{3}}e^{i\theta}\sqrt{3}|3\rangle\right) =$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left(\frac{1}{\sqrt{3}}(2e^{i\theta}-1)|1\rangle - \sqrt{2}e^{i\theta}|3\rangle\right)$$

$$\langle p_x\rangle = \langle\psi(0)|\hat{p}_x|\psi(0)\rangle = \left(\frac{1}{\sqrt{3}}\langle 0| + \sqrt{\frac{2}{3}}e^{-i\theta}\langle 2|\right)\left(-i\sqrt{\frac{m\omega\hbar}{2}}\right)\left(\frac{1}{\sqrt{3}}(2e^{i\theta}-1)|1\rangle - \sqrt{2}e^{i\theta}|3\rangle\right) = 0$$

$$\langle p_x^2\rangle = (\langle\psi(0)|\hat{p}_x)(\hat{p}_x|\psi(0)\rangle) =$$

$$= +i\sqrt{\frac{m\omega\hbar}{2}}\left(\frac{1}{\sqrt{3}}(2e^{-i\theta}-1)\langle 1| - \sqrt{2}e^{-i\theta}\langle 3|\right)\left(-i\sqrt{\frac{m\omega\hbar}{2}}\right)\left(\frac{1}{\sqrt{3}}(2e^{i\theta}-1)|1\rangle - \sqrt{2}e^{i\theta}|3\rangle\right) =$$

$$= \frac{m\omega\hbar}{2}\left(\frac{1}{3}(2e^{-i\theta}-1)(2e^{i\theta}-1)+2\right) = \frac{m\omega\hbar}{2}\left(\frac{1}{3}(4-2e^{-i\theta}-2e^{i\theta}+1)+2\right) =$$

$$= \frac{m\omega\hbar}{2}\left(\frac{1}{3}\left(5-2\cdot 2\underbrace{\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}_{\cos\theta}\right)+2\right) = \frac{m\omega\hbar}{2}\left(\frac{1}{3}(5-4\cos\theta)+2\right) =$$

$$= \frac{m\omega\hbar}{6}(5-4\cos\theta+6) = \boxed{\frac{m\omega\hbar}{6}(11-4\cos\theta)}$$

(-1)

$$\Delta p = \sqrt{\langle p_x^2\rangle - \langle p_x\rangle^2} = \sqrt{\frac{m\omega\hbar}{6}(11-4\cos\theta)} = \sqrt{\frac{5m\omega\hbar}{2}}$$

$$\frac{11-4\cos\theta}{6} = \frac{5}{2} \Rightarrow 11-4\cos\theta = 15$$

$$4\cos\theta = -4 \Rightarrow \cos\theta = -1$$

$$\theta = \pi$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}\overbrace{e^{i\pi}}^{-1}|2\rangle$$

$\theta = \pi$

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}\,e^{i\pi}|2\rangle$$

$$\boxed{|\psi(0)\rangle = \frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}\,|2\rangle}$$

$$\langle x|\psi(0)\rangle = \sqrt{\tfrac{1}{3}}\underbrace{\langle x|0\rangle}_{\psi_0(x)} - \sqrt{\tfrac{2}{3}}\underbrace{\langle x|2\rangle}_{\psi_2(x)}$$

b) **(Bonus 3 Points)** What is the physical meaning of the following integral?

$$\int_{-\infty}^{\infty}\Bigg(\sqrt{\frac{1}{3}}\underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega x^2}{2\hbar}}}_{\psi_0(x)}-\sqrt{\frac{2}{3}}\underbrace{\left(\frac{m\omega}{4\pi\hbar}\right)^{\frac{1}{4}}\left(2\frac{m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega x^2}{2\hbar}}}_{\psi_2(x)}\Bigg)$$
$$\times\underbrace{\left(-\frac{\hbar^2}{2m}\right)\frac{d^2}{dx^2}}_{\hat{p}_x^2}\underbrace{\left(\sqrt{\frac{1}{3}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega x^2}{2\hbar}}-\sqrt{\frac{2}{3}}\left(\frac{m\omega}{4\pi\hbar}\right)^{\frac{1}{4}}\left(2\frac{m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega x^2}{2\hbar}}\right)}_{\langle x|\psi(0)\rangle}dx =$$

$$= \langle \psi(0)|\frac{\hat{p}_x^2}{2m}|\psi(0)\rangle = \left\langle \frac{\hat{p}_x^2}{2m}\right\rangle = \frac{m\omega\hbar}{2m\,6}(11-4\cos\pi)$$

$$= \frac{m\omega\hbar}{2m\,6}(11+4) = \boxed{\frac{5m\omega\hbar}{2m\,2}} = \frac{5\omega\hbar}{4}$$

$$\frac{\langle\hat{p}_x^2\rangle}{2m} = \langle K\rangle$$

c) **(Bonus 2 Points)** What is the value of this integral? Hint: please do not spend time on integration

$$\left\langle\frac{\hat{p}_x^2}{2m}\right\rangle = \langle\psi(0)|\frac{\hat{p}_x^2}{2m}|\psi(0)\rangle = \frac{1}{2m}\int_{-\infty}^{\infty}\underbrace{\langle\psi(0)|x\rangle}_{\psi^*(0)}\langle x|\hat{p}_x^2|\psi(0)\rangle\,dx$$

$$1 = \int |x\rangle\langle x|\,dx$$

$$\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\langle x|\psi(0)\rangle$$

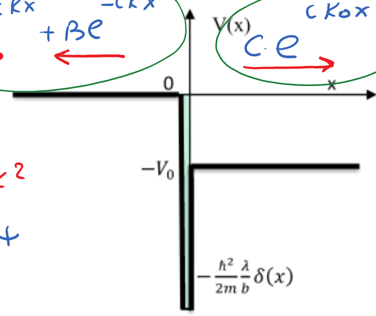
$$= \int\left(\psi^*(0)\,-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\,\psi(0)\right)dx$$

Problem 4 (25 points)

Consider a one-dimensional potential with a step function component and an attractive delta function component just at the edge:

$$V(x) = -\frac{\hbar^2}{2m}\frac{\lambda}{b}\delta(x) + \begin{cases} 0 & x<0 \\ -V_0 & x>0 \end{cases}$$

$Ae^{ikx} + Be^{-ikx}$ $\quad$ $C e^{ik_0x}$



1) $x<0 \quad V=0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \quad k^2$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

Find the transmission coefficient for particles incident from the left with energy $E > 0$.

$$j_{inc} = \frac{\hbar k}{m}|A|^2$$

2) $x>0 \quad V=-V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \qquad \frac{d^2\psi}{dx^2} = -\frac{2m(E+V_0)}{\hbar^2}\psi \quad k_0^2$$

$$\psi(x) = Ce^{ik_0x} + \cancel{De^{-ik_0x}} \qquad D=0$$

$$j_{TR} = \frac{\hbar k_0}{m}|C|^2$$

$$T = \frac{j_{TR}}{j_{inc}} = \frac{\cancel{\hbar} k_0 |C|^2 \cdot \cancel{m}}{\cancel{m}\cdot \cancel{\hbar} k|A|^2} = \frac{k_0|C|^2}{k|A|^2}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\int_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2}dx = \frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon}(V(x)-E)\psi\, dx$$

$$\lim_{\varepsilon\to 0}\left.\frac{d\psi}{dx}\right|_{x=\varepsilon} - \left.\frac{d\psi}{dx}\right|_{x=-\varepsilon} = \frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon}V(x)\psi(x)\,dx - \frac{2m}{\hbar}E\int_{-\varepsilon}^{\varepsilon}\psi(x)\,dx \to 0$$

$$-\frac{\hbar^2}{2m}\frac{\lambda}{b}\delta(x)$$

$$\frac{\cancel{2m}}{\cancel{\hbar^2}}\cdot\left(-\frac{\cancel{\hbar^2}}{\cancel{2m}}\frac{\lambda}{b}\right)\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi(x)\,dx$$

$$\frac{2m}{\hbar^2}\cdot\left(-\frac{\hbar^2}{2m}\frac{\lambda}{b}\right)\int_{-\varepsilon}\delta(x)\psi(x)\,dx$$

$$\boxed{\left.\frac{d\psi}{dx}\right|_{0^+}-\left.\frac{d\psi}{dx}\right|_{0^-} = -\frac{\lambda}{b}\psi(0)}$$

Match Wave function at $x=0$

1) Continuity : $A+B=C$

2) $\left.\frac{d\psi}{dx}\right|_{0^+}-\left.\frac{d\psi}{dx}\right|_{0^-}=-\frac{\lambda}{b}\psi(0)$: $C\cdot i k_0-(Aik-Bik)=-\frac{\lambda}{b}C$ $(A+B)$

$B=C-A$

$$i\left(Ck_0-Ak+Bk\right)=-\frac{\lambda}{b}C$$

$$(-i)\,i\left(Ck_0-Ak+(C-A)k\right)=-\frac{\lambda}{b}C\cdot(-i)$$

$$Ck_0-2Ak+Ck=\frac{\lambda i}{b}C$$

$$C(k_0+k)-\frac{\lambda}{b}iC=2Ak$$

$$C\left((k_0+k)-\frac{\lambda i}{b}\right)=2Ak$$

$$C=\frac{2Ak}{(k_0+k)-\frac{\lambda}{b}i}$$

$$\frac{C}{A}=\frac{2k}{(k_0+k)-\frac{\lambda}{b}i}$$

$$\left|\frac{C}{A}\right|^2=\frac{2k}{(k_0+k)-\frac{\lambda}{b}i}\cdot\frac{2k}{(k_0+k)+\frac{\lambda}{b}i}=\frac{4k^2}{(k_0+k)^2+\left(\frac{\lambda}{b}\right)^2}$$

$$T_{TR} = \frac{K_0}{K}\left|\frac{C}{A}\right|^2 = \frac{K_0}{K}\cdot\frac{4K^2}{(K_0+K)^2+\left(\frac{\lambda}{\beta}\right)^2} = \boxed{\frac{4K_0K}{(K_0+K)^2+\left(\frac{\lambda}{\beta}\right)^2}}$$