Summary of previous Lecture

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t)=i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

probability current

$$j_x=\frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right)\Rightarrow\frac{\partial(\psi^*\psi)}{\partial t}=-\frac{\partial j_x}{\partial x}$$

$$\frac{d}{dt}\int_a^b\psi^*\psi\,dx=-j_x(b,t)+j_x(a,t)$$

Stationary states:

$$\Psi(x,t) = e^{-\frac{iEt}{\hbar}}\,\psi(x) \qquad \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\,\psi(x)$$

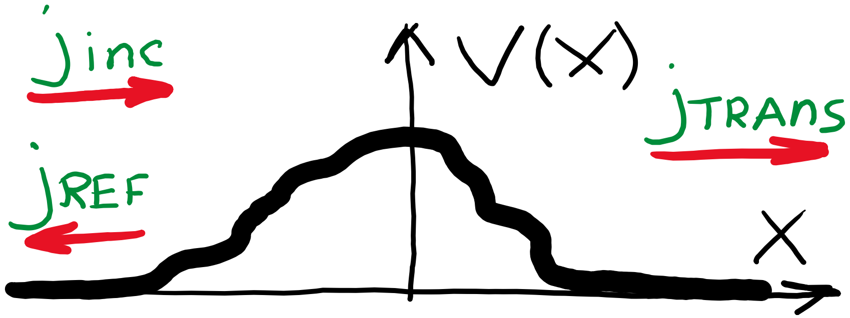
$$\frac{\partial}{\partial t}\left(\Psi^*(x,t)\,\Psi(x,t)\right) = 0 = -\frac{\partial j_x}{\partial x} \Rightarrow j_x = \text{const}$$

If $\psi(x)$ is real (or $e^{i\theta}$ times real function) then $j_x = 0$

For x interval where $V(x) = \text{const} < E$:

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

$$j_x = \frac{\hbar k}{m}|A|^2 - \frac{\hbar k}{m}|B|^2$$



$$T = \frac{j_{Trans}}{j_{inc}} \qquad R = \frac{j_{Ref}}{j_{inc}}$$

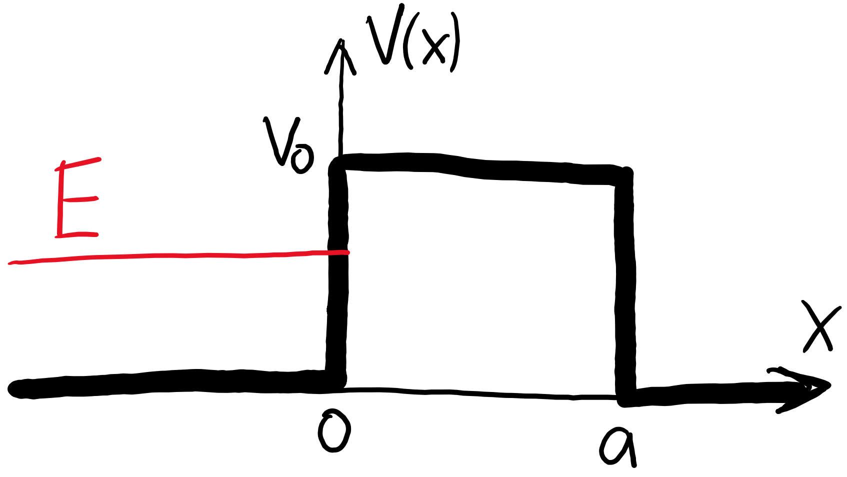
$$T + R = 1$$

Tunneling

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dX^2}+V(X)\right]\psi(X)=E\psi(X)$$

$$V(x)=\begin{cases}0 & x<0\\ V_0 & 0<x<a\\ 0 & x>a\end{cases}$$

$0<E<V_0$



1) $X<0 \quad V=0$

$$\frac{d^2\psi}{dx^2}=-\frac{2mE}{\hbar^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=-k^2\psi \Rightarrow \psi=Ae^{ikx}+Be^{-ikx}$$

k^2

$$j_{inc}=\frac{\hbar k}{m}|A|^2 \qquad j_{REF}=\frac{\hbar k}{m}|B|^2$$

2) $0<X<a \quad V=V_0$

$$\frac{d^2\psi}{dx^2}=\frac{2m(V_0-E)}{\hbar^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=q^2\psi \Rightarrow \psi=Fe^{qx}+Ge^{-qx}$$

q^2

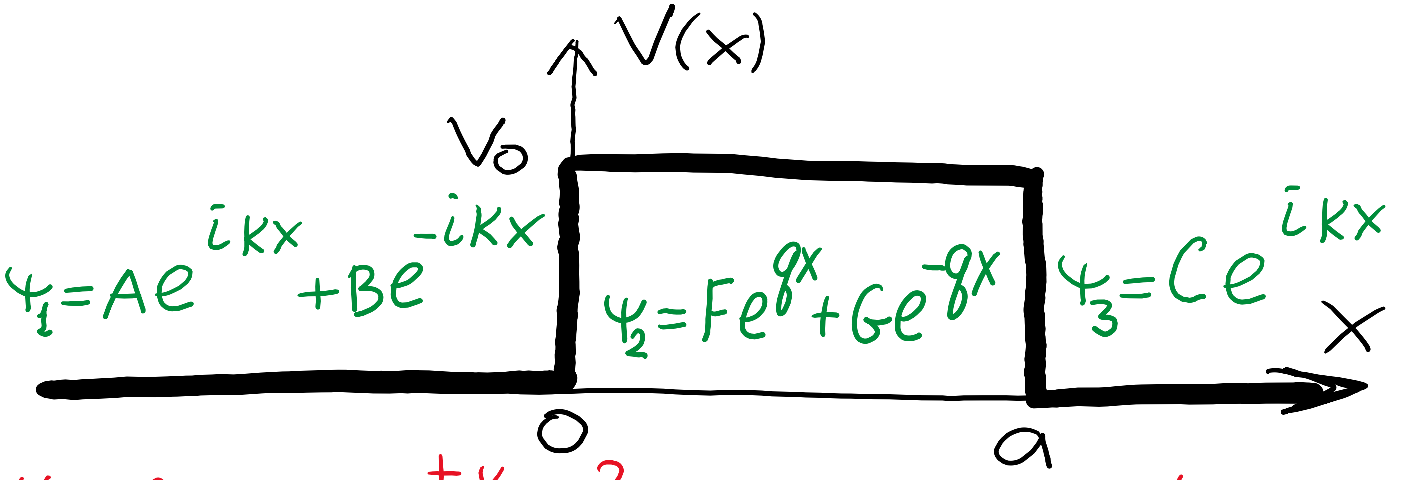
3) $X>a \quad V=0$

Same as for $X<0$: $\quad \frac{d^2\psi}{dx^2}=-k^2\psi \Rightarrow \psi=Ce^{ikx}+De^{-ikx}$

$$j_{TRANS}=\frac{\hbar k}{m}|C|^2$$

$$T = \frac{j_{TRANS}}{j_{inc}} = \frac{|C|^2}{|A|^2}$$

$$R = \frac{j_{REF}}{j_{inc}} = \frac{|B|^2}{|A|^2}$$



$$j_{inc} = \frac{\hbar k}{m}|A|^2 \qquad j_{REF} = \frac{\hbar k}{m}|B|^2 \qquad j_{TRANS} = \frac{\hbar k}{m}|C|^2$$

1) $x=0$

Continuity: $\psi_1(0) = \psi_2(0) \Rightarrow A + B = F + G$

Smoothness: $\left.\frac{d\psi_1}{dx}\right|_{x=0} = \left.\frac{d\psi_2}{dx}\right|_{x=0} \Rightarrow ikA - ikB = qF - qG$

2) $x=a$

Continuity: $\psi_2(a) = \psi_3(a) \Rightarrow Fe^{qa} + Ge^{-qa} = Ce^{ika}$

Smoothness: $\left.\frac{d\psi_2}{dx}\right|_{x=a} = \left.\frac{d\psi_3}{dx}\right|_{x=a} \Rightarrow qFe^{qa} - qGe^{-qa} = ikCe^{ika}$

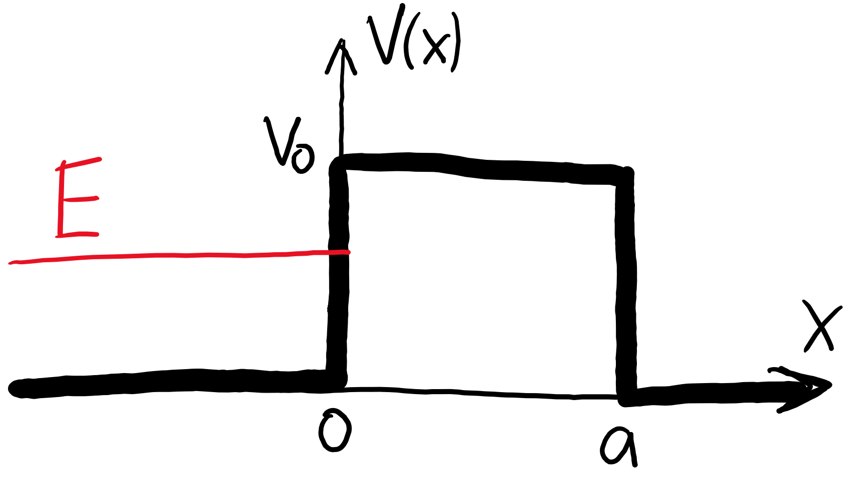
$$T = \frac{j_{TRANS}}{j_{inc}} = \frac{|C|^2}{|A|^2} = \frac{1}{1 + \left(\frac{K^2+q^2}{2Kq}\right)^2 \sinh^2(qa)}$$

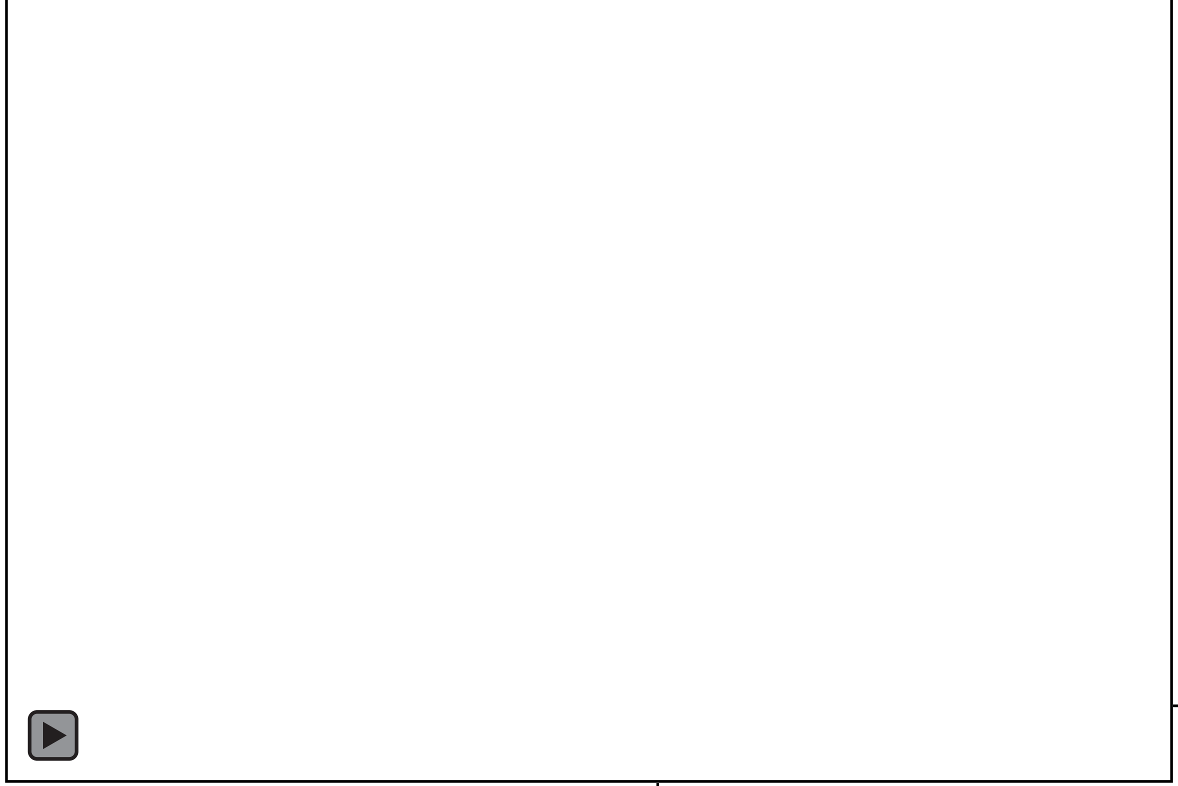
$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

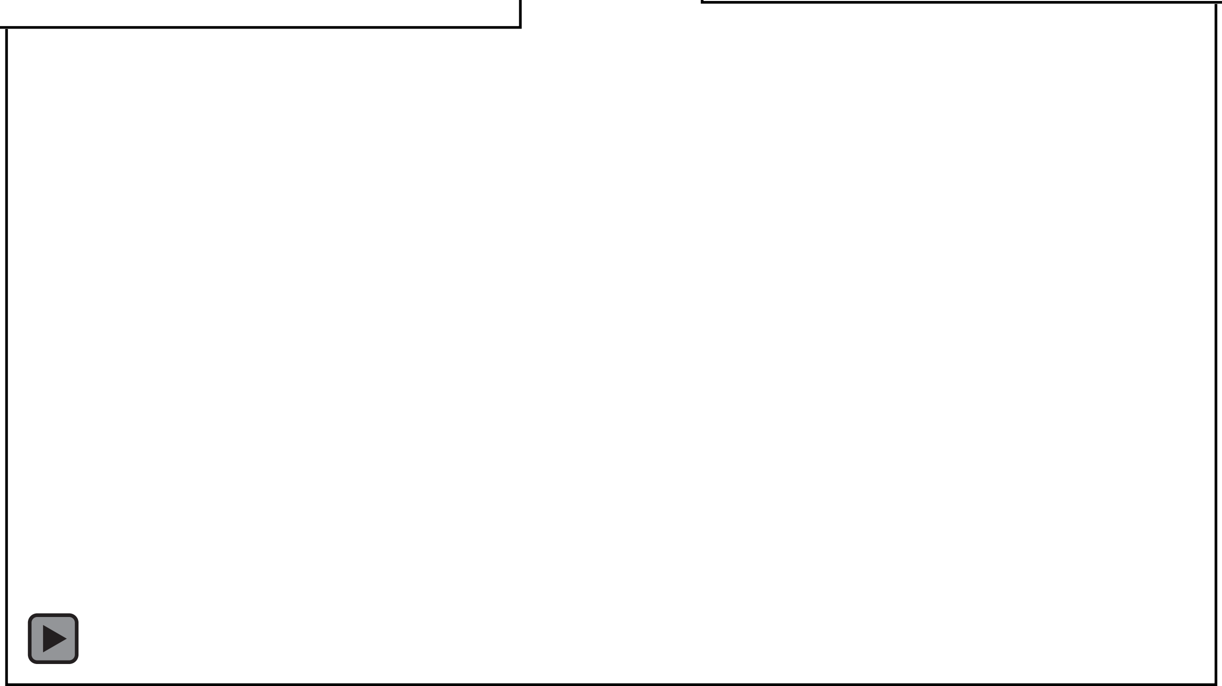
$$\sinh(qa) = \frac{e^{qa} - e^{-qa}}{2} \approx \frac{e^{qa}}{2}$$

$$q = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$T \xrightarrow[qa \gg 1]{} \left(\frac{4Kq}{K^2+q^2}\right)^2 e^{-2qa}$$



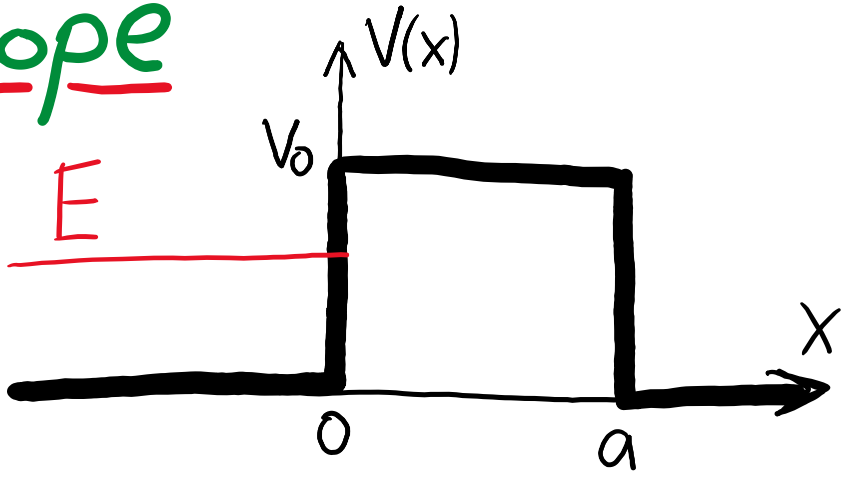


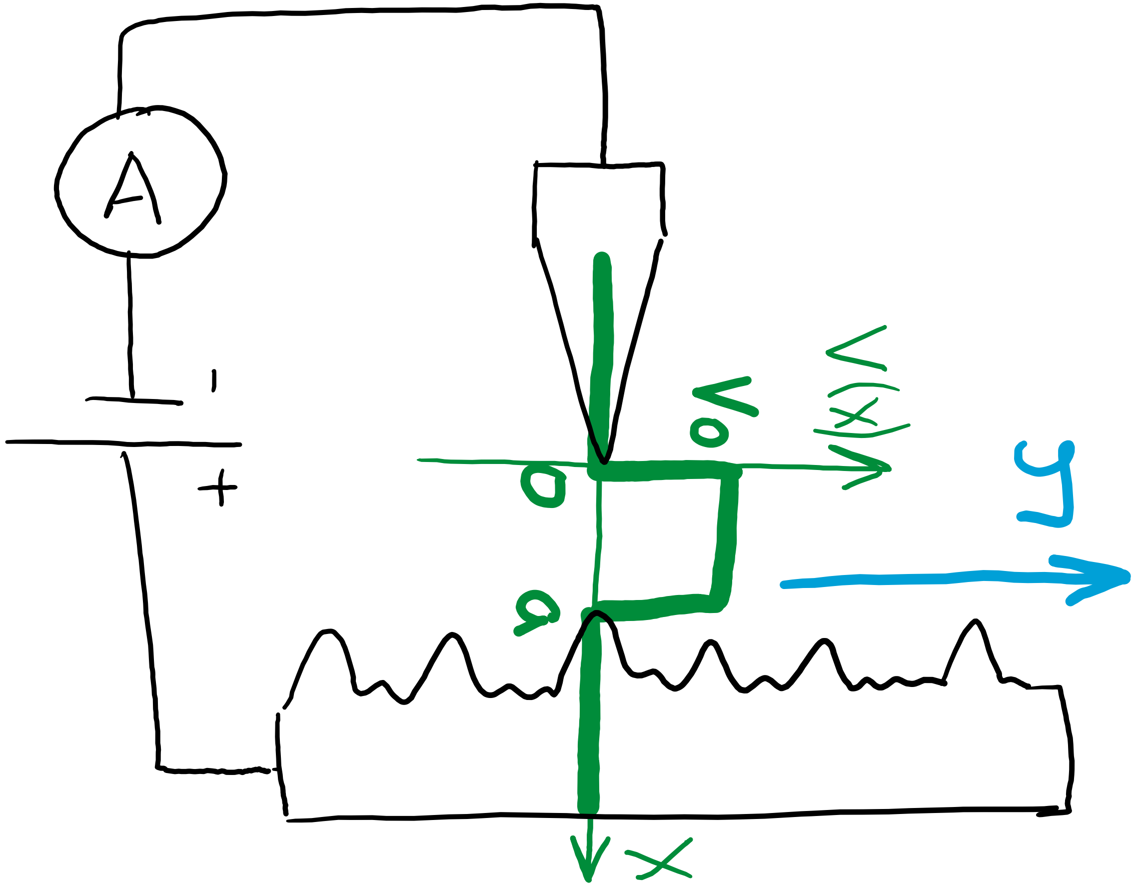


Scanning tunneling microscope

$$T \sim e^{-2qa}$$

$$q = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

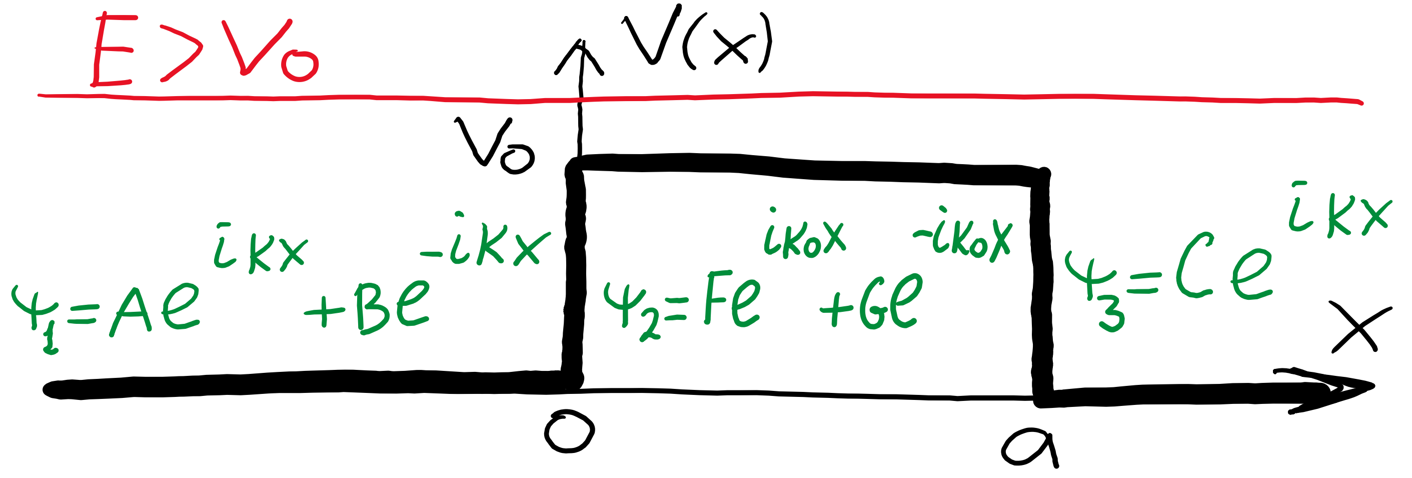




$$I \sim T \sim e^{-2qa}$$

$$I(y) \sim e^{-2qa(y)}$$

Over-the-barrier scattering



$0 < x < a$ $\quad V = V_0$

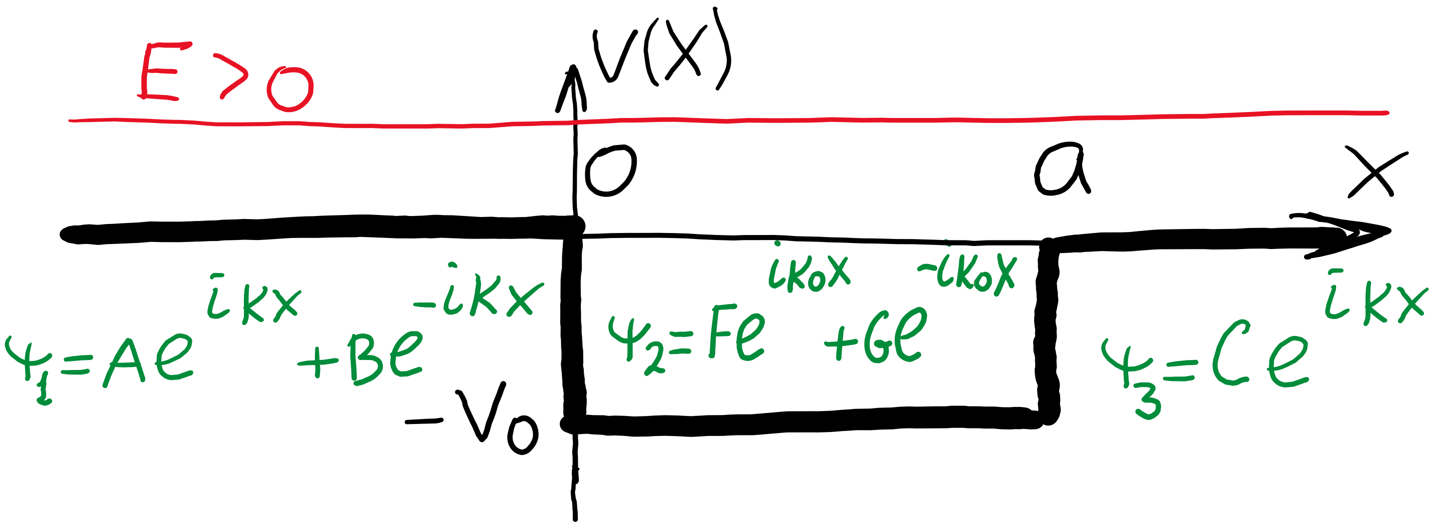
$$\frac{d^2\psi}{dx^2} = -\overbrace{\frac{2m(E-V_0)}{\hbar^2}}^{k_0^2}\psi$$

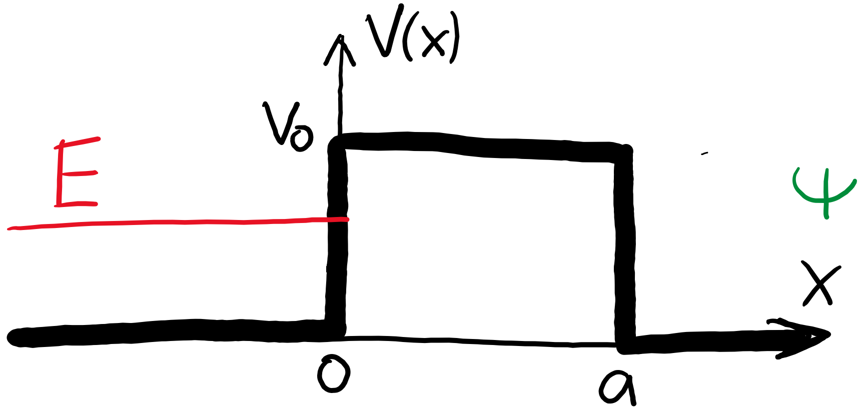
$$\frac{d^2\psi}{dx^2} = -k_0^2\psi \Rightarrow \psi_2 = Fe^{ik_0x} + Ge^{-ik_0x} \quad \text{where } k_0 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$0 < x < a$ $\quad V = -V_0$

$$\frac{d^2\psi}{dx^2} = -\overbrace{\frac{2m(E+V_0)}{\hbar^2}}^{k_0^2}\psi$$

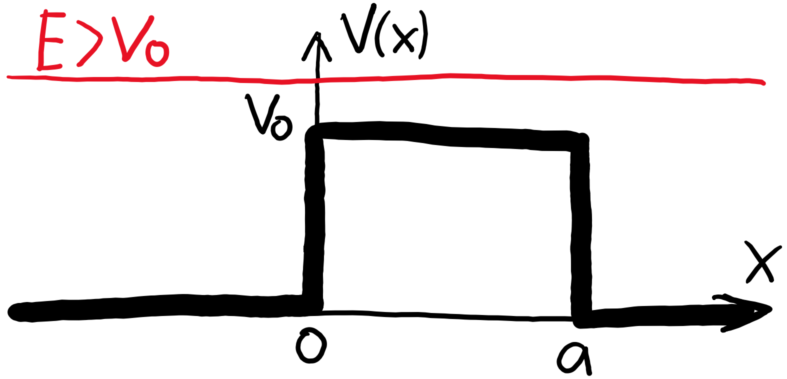
$$\text{where } k_0 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

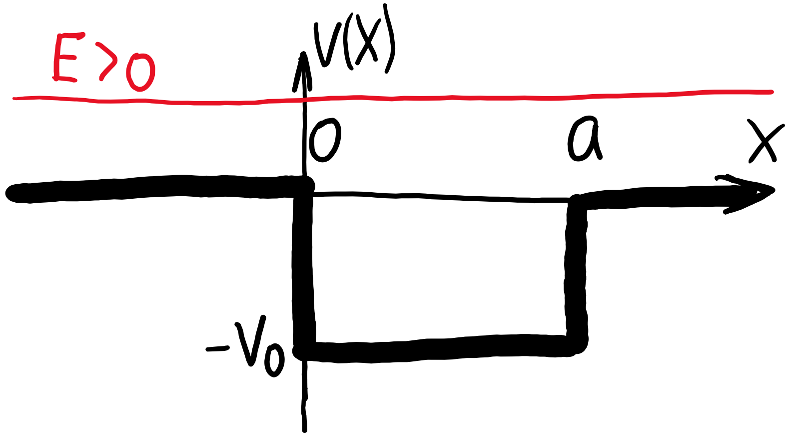




$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Fe^{qx} + Ge^{-qx} & 0<x<a \\ Ce^{ikx} & x>a \end{cases}$$

$$T = \frac{1}{1 + \left(\frac{k^2+q^2}{2kq}\right)^2 \sinh^2(qa)}$$





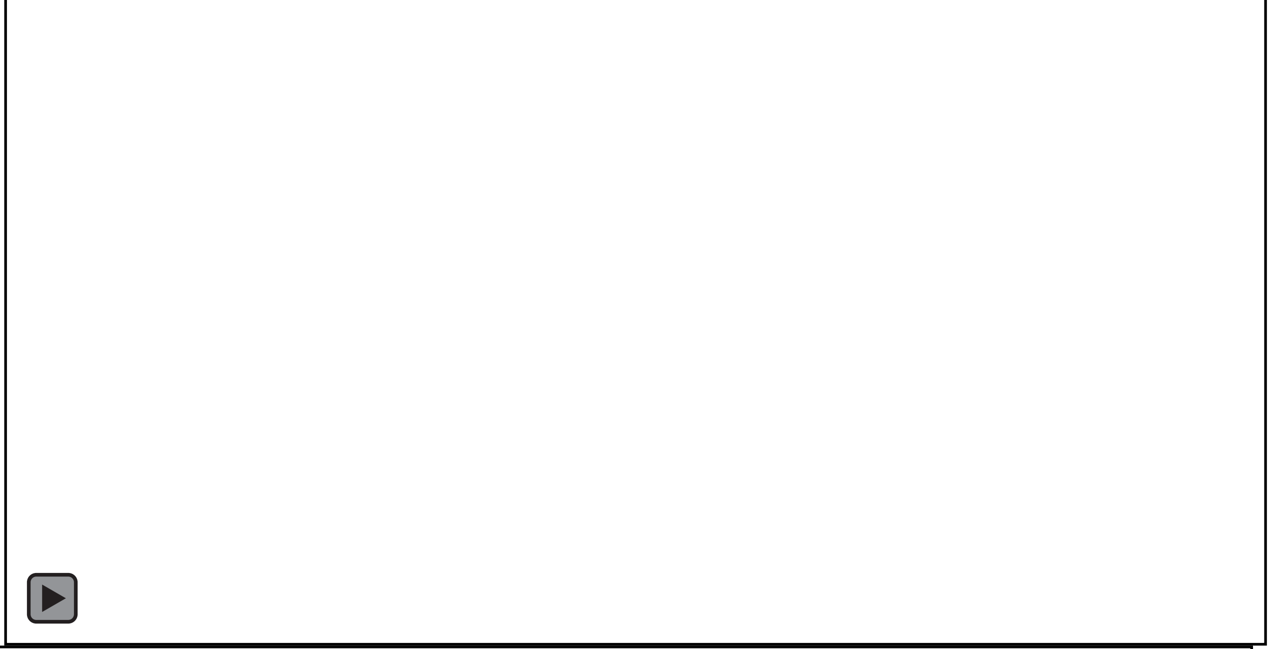
$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Fe^{ik_0x} + Ge^{-ik_0x} & 0<x<a \\ Ce^{ikx} & x>a \end{cases}$$

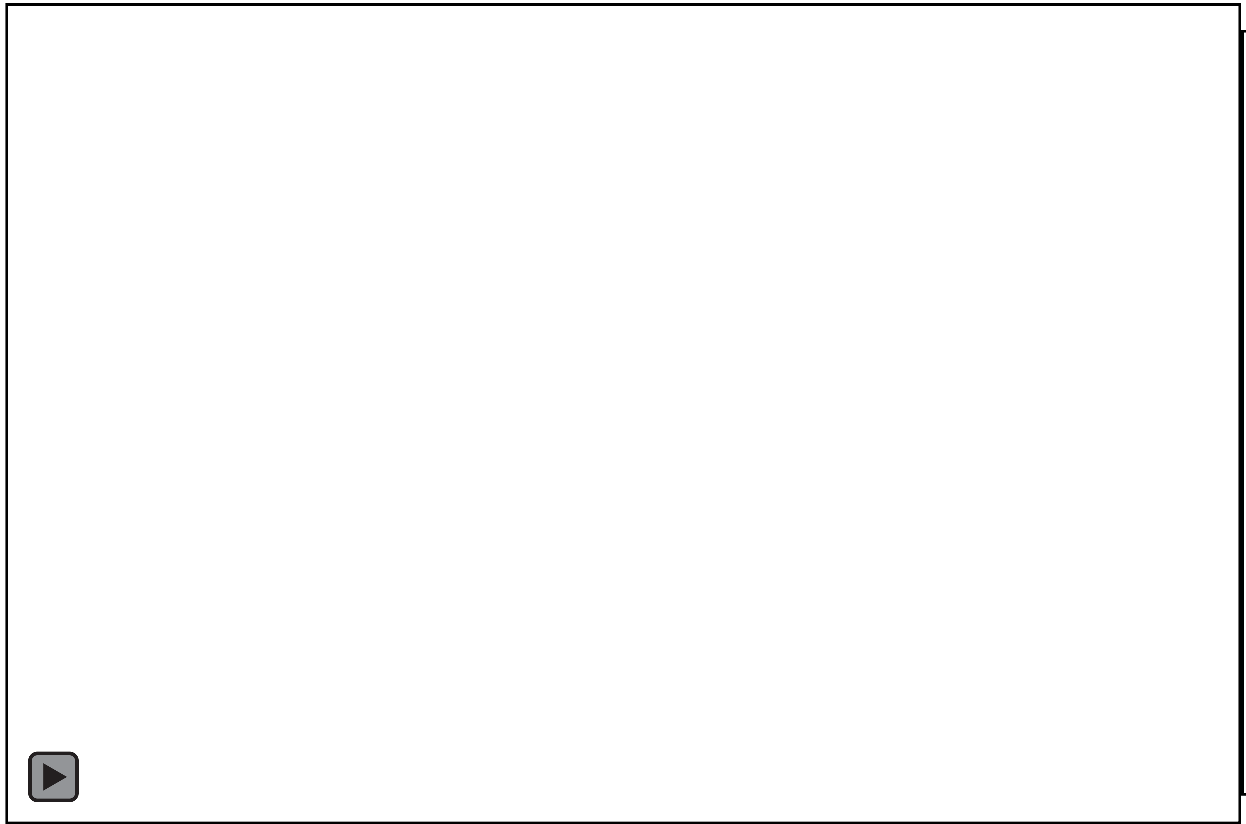
$$q \longrightarrow ik_0$$

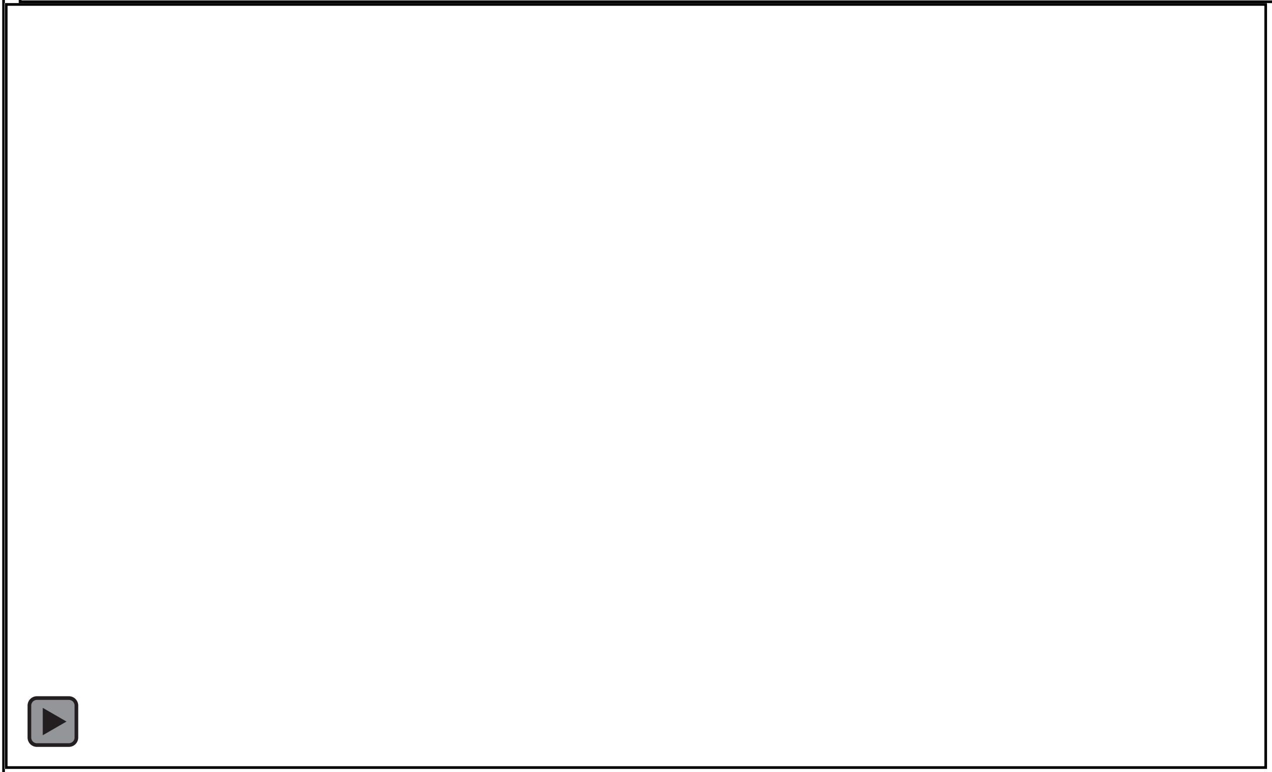
$$\sinh(qa) \longrightarrow \sinh(ik_0a) = i\,\frac{e^{ik_0a} - e^{-ik_0a}}{2i} = i\sin k_0a$$

$$T = \frac{1}{1 + \left(\frac{k^2-k_0^2}{2kk_0}\right)^2 \sin^2(k_0a)}$$

$$T = \frac{1}{1 + \left(\frac{k^2 - k_0^2}{2k k_0}\right)^2 \sin^2(k_0 a)}$$



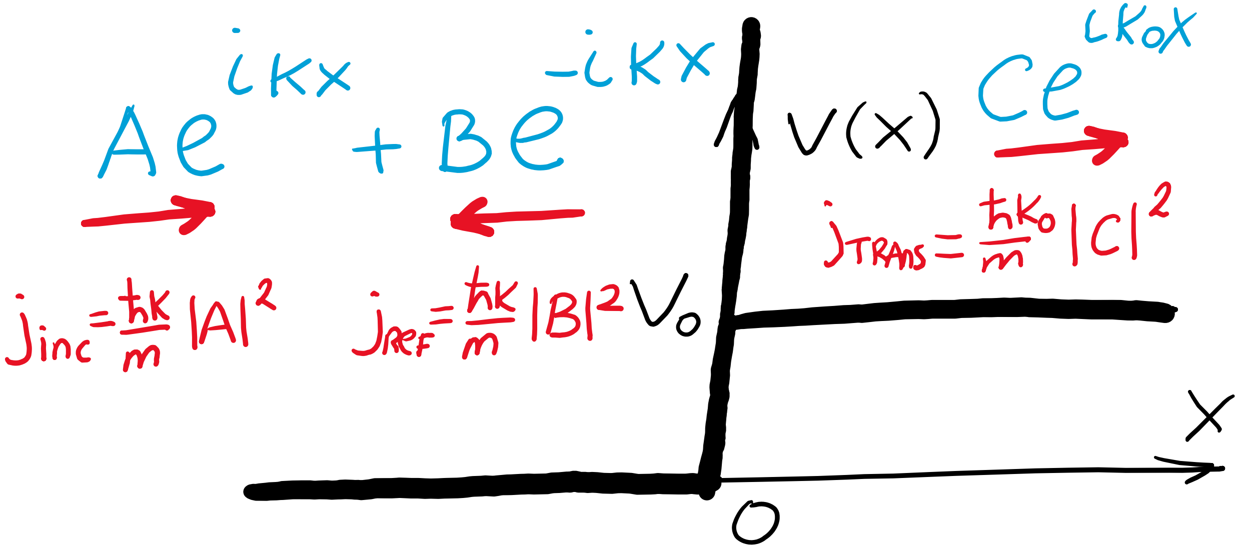




Scattering from δ-function barrier

$$V(x)=\begin{cases}0 & x<0\\ V_0 & x>0\end{cases}+\frac{\lambda}{6}\frac{\hbar^2}{2m}\delta(x)$$

$E>V_0$



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi$$

1) $x<0$ $\quad V=0$

$$\frac{d^2\psi}{dx^2}=-\underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=-k^2\psi \Rightarrow \psi(x)=Ae^{ikx}+Be^{-ikx}$$

2) $x>0$ $\quad V=V_0$

$$\frac{d^2\psi}{dx^2}=-\underbrace{\frac{2m}{\hbar^2}(E-V_0)}_{k_0^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=-k_0^2\psi \Rightarrow \psi(x)=Ce^{ik_0x}+De^{-ik_0x}$$

$j_{TRANS}=\frac{\hbar k_0}{m}|C|^2$

$$\int_{-\varepsilon}^{\varepsilon} \frac{d^2\psi}{dx^2}dx = \int_{-\varepsilon}^{\varepsilon} \frac{2m}{\hbar^2}\left[V(x)-E\right]\psi\, dx$$

$$\lim_{\varepsilon\to 0} \left(\frac{d\psi}{dx}\right)_{\varepsilon} - \left(\frac{d\psi}{dx}\right)_{-\varepsilon} = \frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon} V(x)\psi\, dx - \underbrace{\frac{2mE}{\hbar^2}\int_{-\varepsilon}^{\varepsilon}\psi\, dx}_{0}$$

$$\left(\frac{d\psi}{dx}\right)_{0^+} - \left(\frac{d\psi}{dx}\right)_{0^-} = \frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} V(x)\psi\, dx = \frac{\lambda}{\beta}\psi(0)$$

$$V(x) = \begin{cases} 0 & x<0 \\ V_0 & x>0 \end{cases} + \frac{\lambda}{\beta}\frac{\hbar^2}{2m}\delta(x)$$

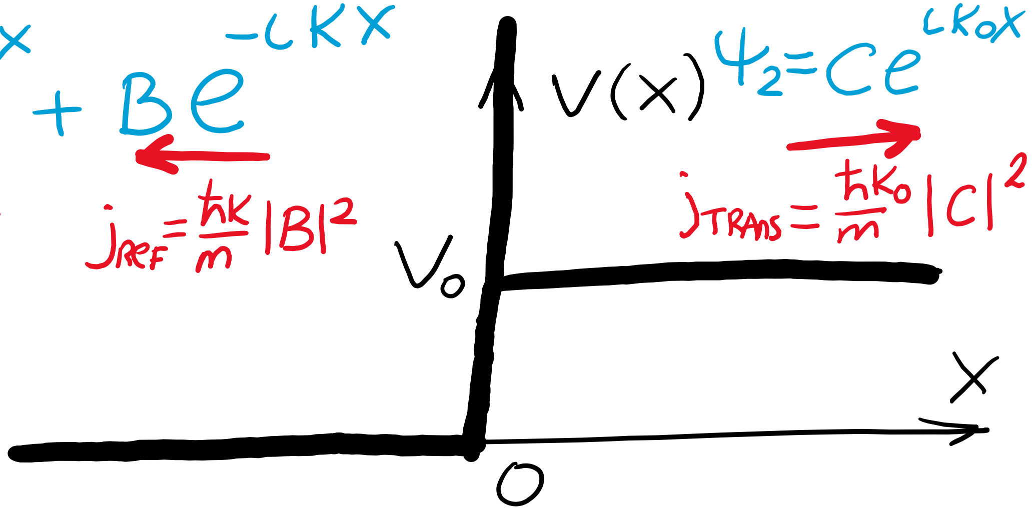
$$T = \frac{j_{Trans}}{j_{inc}} = \frac{k_0|C|^2}{k|A|^2}$$

$$\psi_1 = Ae^{ikx} + Be^{-ikx}$$

$$j_{inc} = \frac{\hbar k}{m}|A|^2 \qquad j_{REF} = \frac{\hbar k}{m}|B|^2$$

$$\psi_2 = Ce^{ik_0x}$$

$$j_{Trans} = \frac{\hbar k_0}{m}|C|^2$$



Match at x=0

1) Continuity $\psi_1(0) = \psi_2(0) \Rightarrow A + B = C$

2) $\left.\frac{d\psi_2}{dx}\right|_{x=0} - \left.\frac{d\psi_1}{dx}\right|_{x=0} = \frac{\lambda}{\beta}\psi_2(0) \Rightarrow ik_0C - ikA + ikB = \frac{\lambda}{\beta}C$

$$\frac{C}{A} = \frac{2k}{k+k_0+i\frac{\lambda}{\beta}} \Rightarrow \left|\frac{C}{A}\right|^2 = \frac{2k}{k+k_0+i\frac{\lambda}{\beta}} \cdot \frac{2k}{k+k_0-i\frac{\lambda}{\beta}} = \frac{4k^2}{(k+k_0)^2+\left(\frac{\lambda}{\beta}\right)^2}$$

$$T = \frac{k_0}{k}\left|\frac{C}{A}\right|^2 = \frac{4k_0k}{(k+k_0)^2+\left(\frac{\lambda}{\beta}\right)^2}$$