$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

Unconstrained particle

$\Downarrow$

Continuous energy spectrum

Harmonic potential:

$$V(x)=\frac{1}{2}m\omega^2x^2$$

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right) \qquad n=0,1,$$

as $\omega\to 0$ $\quad \Delta E\to 0$

Infinite square Well:

$$V(x)=\begin{cases}0 & 0<x<L\\ \infty & \text{elsewhere}\end{cases}$$

$$E_n=\frac{\hbar^2\pi^2n^2}{2mL^2} \qquad n=1,2,$$

as $L\to\infty$ $\quad \Delta E\to 0$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi(x,t)=i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

$$\frac{\partial\psi}{\partial t}=\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi\right]$$

$$\frac{\partial\psi^*}{\partial t}=-\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}+V(x)\psi^*\right]$$

$$\frac{\partial(\psi^*\psi)}{\partial t}=\psi\frac{\partial\psi^*}{\partial t}+\psi^*\frac{\partial\psi}{\partial t}=-\frac{\psi}{i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}+V(x)\psi^*\right]+\frac{\psi^*}{i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi\right]$$

$$=\frac{\hbar}{2mi}\psi\frac{\partial^2\psi^*}{\partial x^2}-\frac{V(x)}{i\hbar}\psi\psi^*-\frac{\hbar}{2mi}\psi^*\frac{\partial^2\psi}{\partial x^2}+\frac{V(x)}{i\hbar}\psi^*\psi=\frac{\hbar}{2mi}\left(\psi\frac{\partial^2\psi^*}{\partial x^2}-\psi^*\frac{\partial^2\psi}{\partial x^2}\right)$$

$$\frac{\partial(\psi^*\psi)}{\partial t} = \frac{\hbar}{2mi}\left(\psi\frac{\partial^2\psi^*}{\partial x^2} - \psi^*\frac{\partial^2\psi}{\partial x^2}\right)$$

$$j_x = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)$$

$$\frac{\partial j_x}{\partial x} = \frac{\hbar}{2mi}\left(\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x} + \psi^*\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi}{\partial x}\cdot\frac{\partial\psi^*}{\partial x} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right) = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)$$

$$\frac{\partial(\psi^*\psi)}{\partial t} = -\frac{\partial j_x}{\partial x}$$

$$P(a<x<b) = \int_a^b \psi^*\psi\, dx$$

$$\frac{d}{dt}\int_a^b \psi^*\psi\, dx = \int_a^b \frac{\partial(\psi^*\psi)}{\partial t}\, dx = \int_a^b -\frac{\partial j_x}{\partial x}\, dx = -j_x(b,t) + j_x(a,t)$$

$$\frac{d}{dt}\int^{b}\psi^*\psi\,dx = -J(b,t) + J(a,t)$$

Probability Current

$$j_x = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)$$

Assume $\psi(x,t) = e^{i\theta}\varphi(x,t)$ where φ is Real function

Then $\psi^*(x,t) = e^{-i\theta}\varphi(x,t)$

$$\psi^*\frac{d\psi}{dx} - \psi\frac{\partial\psi^*}{\partial x} = e^{-i\theta}\varphi\, e^{i\theta}\frac{\partial\varphi}{\partial x} - e^{i\theta}\varphi\, e^{-i\theta}\frac{\partial\varphi}{\partial x} = 0$$

If $\psi(x,t)$ is Real (or $e^{i\theta}$ times Real function) then $j_x = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi \qquad V(x)=0$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi \Rightarrow \frac{d^2\psi}{dx^2}=-\overbrace{\frac{2mE}{\hbar^2}}^{k^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=-k^2\psi$$

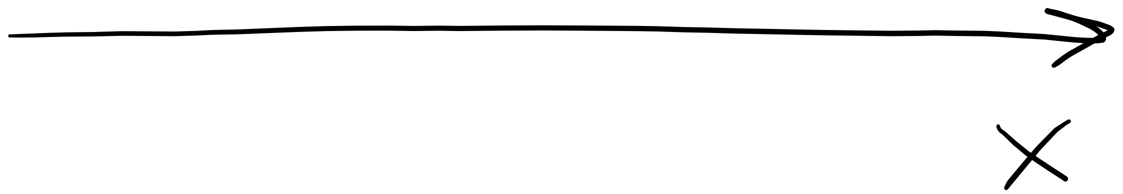
$$\psi(x)=Ae^{ikx}+Be^{-ikx} \Rightarrow \psi^*(x)=A^*e^{-ikx}+B^*e^{ikx}$$

$$j_x=\frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right)=\frac{\hbar}{2mi}\left[\underbrace{\left(A^*e^{-ikx}+B^*e^{ikx}\right)}_{\psi^*}\underbrace{\left(ikAe^{ikx}-ikBe^{-ikx}\right)}_{\frac{\partial\psi}{\partial x}}-\underbrace{\left(Ae^{ikx}+Be^{-ikx}\right)}_{\psi}\underbrace{\left(-ikA^*e^{-ikx}+ikB^*e^{ikx}\right)}_{\frac{\partial\psi^*}{\partial x}}\right]$$

$$=\frac{\hbar}{2mi}\left[ikA^*A-ikA^*Be^{-2ikx}+ikB^*Ae^{2ikx}-ikB^*B+ikA^*A-ikAB^*e^{2ikx}+ikBA^*e^{-2ikx}-ikB^*B\right]$$

$$=\frac{\hbar}{2mi}\left[2ikA^*A-2ikB^*B\right]=\frac{\hbar k}{m}\left(|A|^2-|B|^2\right)$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \Rightarrow j_x = \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)$$



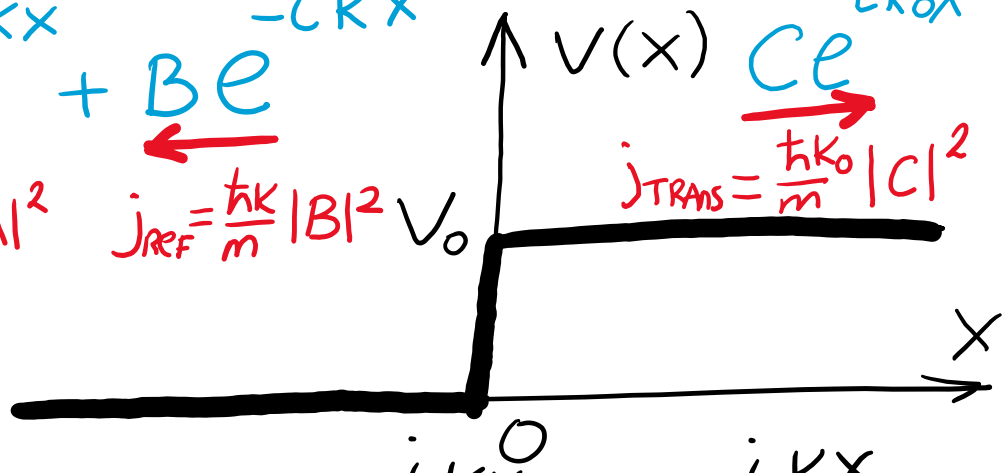
$$j_x = \frac{\hbar k}{m}|A|^2 - \frac{\hbar k}{m}|B|^2$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi$$

$$V(x)=\begin{cases}0 & x<0\\ V_0 & x>0\end{cases}$$

$$\xrightarrow{\;Ae^{ikx}\;} \qquad \xleftarrow{\;+Be^{-ikx}\;} \qquad \xrightarrow{\;Ce^{ik_0x}\;}$$

$$j_{inc}=\frac{\hbar k}{m}|A|^2 \qquad j_{REF}=\frac{\hbar k}{m}|B|^2 \qquad j_{TRANS}=\frac{\hbar k_0}{m}|C|^2$$



1) $x<0 \quad V=0$

$$\frac{d^2\psi}{dx^2}=-\underbrace{\frac{2mE}{\hbar^2}}_{K^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=-K^2\psi \Rightarrow \psi(x)=Ae^{ikx}+Be^{-ikx}$$

2) $x>0 \quad V=V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V_0\psi=E\psi$$

a) $E>V_0$

$$\frac{d^2\psi}{dx^2}=-\underbrace{\frac{2m}{\hbar^2}(E-V_0)}_{k_0^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=-k_0^2\psi \Rightarrow \psi(x)=Ce^{ik_0x}+De^{-ik_0x}$$

$$j_{TRANS}=\frac{\hbar k_0}{m}|C|^2$$

b) $E<V_0$

$$\frac{d^2\psi}{dx^2}=\underbrace{-\frac{2m}{\hbar^2}(E-V_0)}_{q^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=q^2\psi \Rightarrow \psi(x)=Fe^{-qx}+Ge^{qx}$$

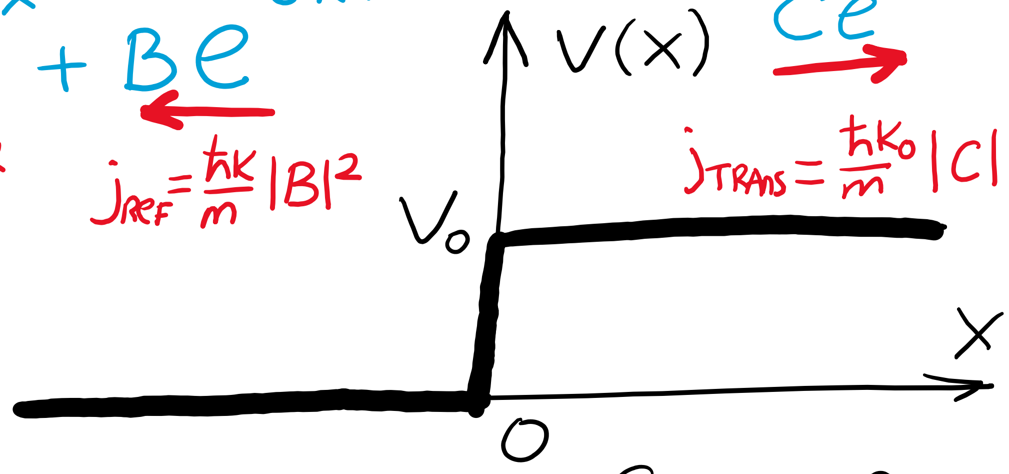
$$j_{TRANS}=0$$

$$T = \frac{j_{Trans}}{j_{inc}} \qquad R = \frac{j_{Ref}}{j_{inc}}$$

$$T + R = 1$$

$$\underset{\rightarrow}{Ae^{ikx}} + \underset{\leftarrow}{Be^{-ikx}} \qquad \underset{\rightarrow}{Ce^{ik_0x}}$$

$$j_{inc} = \frac{\hbar k}{m}|A|^2 \qquad j_{Ref} = \frac{\hbar k}{m}|B|^2 \qquad j_{Trans} = \frac{\hbar k_0}{m}|C|^2$$



a) $E > V_0$

1) Continuity $\quad A + B = C$

2) Smoothness: $\quad ikA - ikB = ik_0C$

$$\Rightarrow \begin{aligned} A + B &= C \\ A - B &= \frac{k_0}{k}C \end{aligned} \Rightarrow 2A = C\left(1 + \frac{k_0}{k}\right) \Rightarrow \frac{C}{A} = \frac{2k}{k + k_0}$$

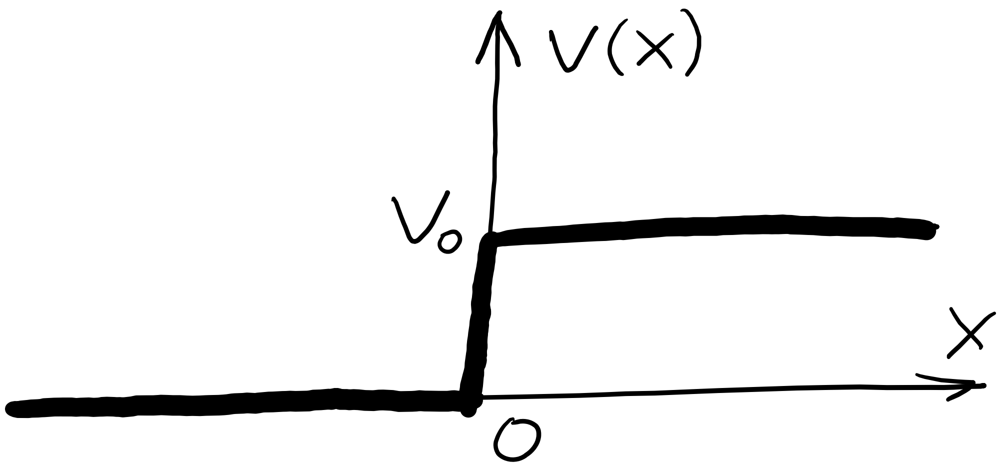
$$\frac{k}{k_0}(A - B) = C = A + B \Rightarrow \frac{B}{A} = \frac{k - k_0}{k + k_0}$$

$$T = \frac{j_{Trans}}{j_{inc}} = \frac{k_0|C|^2}{k|A|^2} = \frac{k_0}{k}\cdot\left|\frac{2k}{k + k_0}\right|^2 = \frac{4k_0k}{(k + k_0)^2}$$

$$R = \frac{j_{Ref}}{j_{inc}} = \frac{|B|^2}{|A|^2} = \frac{(k - k_0)^2}{(k + k_0)^2}$$

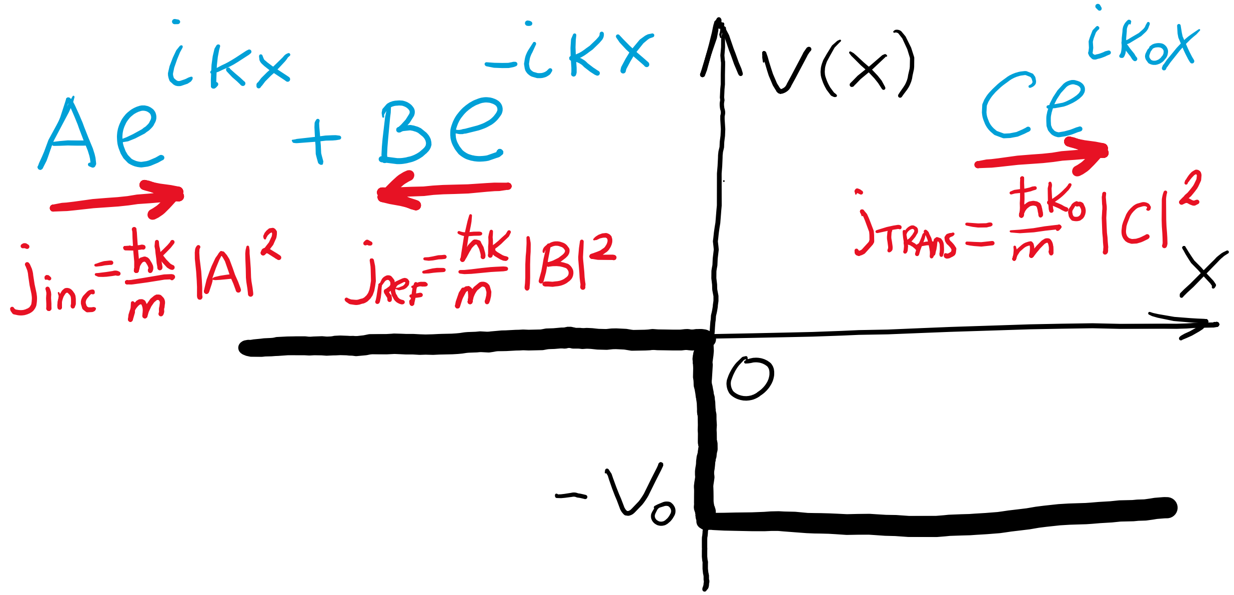
$$T + R = \frac{4k_0k}{(k + k_0)^2} + \frac{(k - k_0)^2}{(k + k_0)^2} = \frac{(k + k_0)^2}{(k + k_0)^2} = 1$$

b) $E < V_0$ $\quad \psi(x) = Fe^{-gx}$ for $x > 0$ $\Rightarrow j_{Trans} = 0 \Rightarrow T = 0 \qquad R = 1$


V(x)
V₀
x
0

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi$$

$$V(x)=\begin{cases}0 & x<0\\ -V_0 & x>0\end{cases}$$



1) $x<0 \quad V=0$

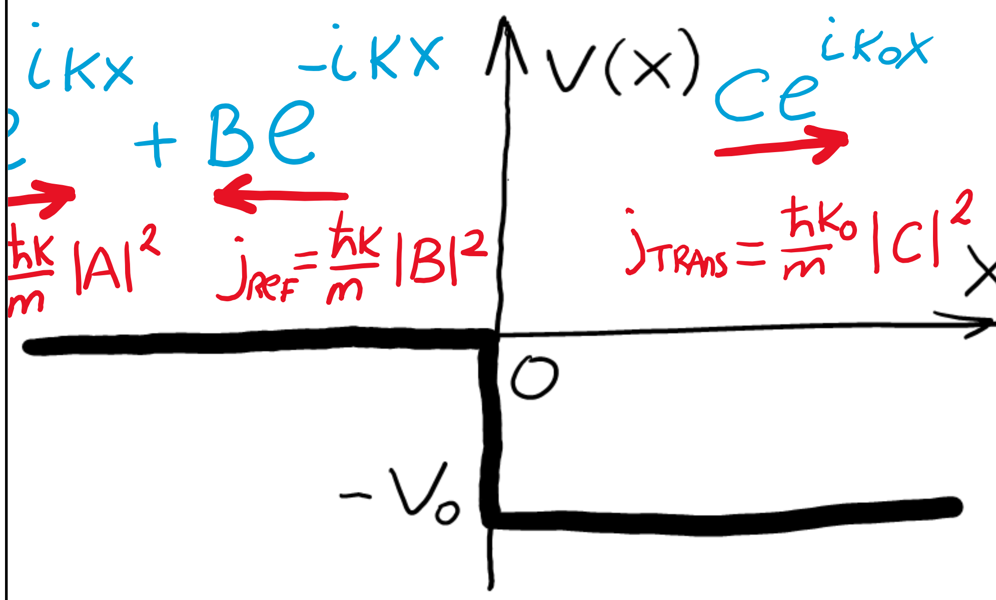
$$\frac{d^2\psi}{dx^2}=-\underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=-k^2\psi \Rightarrow \psi(x)=Ae^{ikx}+Be^{-ikx}$$

2) $x>0 \quad V=-V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}-V_0\psi=E\psi$$

$$\frac{d^2\psi}{dx^2}=-\underbrace{\frac{2m}{\hbar^2}(E+V_0)}_{k_0^2}\psi \Rightarrow \frac{d^2\psi}{dx^2}=-k_0^2\psi \Rightarrow \psi(x)=Ce^{ik_0x}+De^{-ik_0x}$$

$$j_{TRANS}=\frac{\hbar k_0}{m}|C|^2$$



$$T = \frac{j_{TRans}}{j_{inc}} = \frac{k_0|C|^2}{k|A|^2} = \frac{4k_0 k}{(k+k_0)^2}$$

$$R = \frac{j_{Ref}}{j_{inc}} = \frac{|B|^2}{|A|^2} = \frac{(k-k_0)^2}{(k+k_0)^2}$$