## Quantum Mechanics I (PHYS 3143A)

**Test 3** **Name:**

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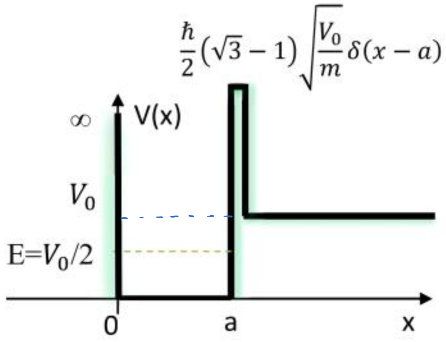
### Problem 1 (60 points)

A particle of mass *m* is in the potential energy well

$$V(x)=\underbrace{\begin{cases}\infty & x<0\\ 0 & 0<x<a\\ V_0 & x>a\end{cases}}_{V_{step}}+\frac{\hbar}{2}\left(\sqrt{3}-1\right)\sqrt{\frac{V_0}{m}}\,\delta(x-a)$$

a) Determine $V_0$ (in terms of given parameters *m* and *a*) such that the ground state energy in this well is $E=V_0/2$.



1) $x<0$

$\psi_1(x)=0$

2) $0<x<a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+0\cdot\psi=E\psi$$

$$\frac{d^2\psi}{dx^2}=-\overset{k^2}{\boxed{\frac{2mE}{\hbar^2}}}\psi \Rightarrow \boxed{\frac{d^2\psi}{dx^2}=\boxed{-}k^2\psi}$$

$$\psi_2(x)=A'e^{ikx}+B'e^{-ikx}=\underline{A\sin kx+B\cos kx}$$

3) $x>a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V_0\psi=E\psi \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=(E-V_0)\psi \Rightarrow \underline{\frac{d^2\psi}{dx^2}=\overset{g^2}{\boxed{\frac{2m(V_0-E)}{\hbar^2}}}\psi}$$

$$\underline{\frac{d^2\psi}{dx^2}=+g^2\psi} \Rightarrow \underline{\psi_3(x)=Ce^{-gx}+\overset{=0}{D}\cancel{e^{gx}}}$$

Match at $x=0$:

Continuity: $\psi_1(0)=\psi_2(0)$ : $0=\underline{A\sin k0}+B\cdot\cos k\cdot 0$

$0=\ 0+B\cdot 1$

$B=0$

$$\psi_2(x)=A\sin kx$$

Match at $x=a$

Continuity: $\underline{\psi_2(a)} = \underline{\psi_3(a)}$ : $\underline{A\sin Ka = C\cdot e^{-ga}}$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

$$\int_{a-\varepsilon}^{a+\varepsilon}\frac{d^2\psi}{dx^2}dx = \frac{2m}{\hbar^2}V(x)\psi - \frac{2mE\psi}{\hbar^2} = \int_{a-\varepsilon}^{a+\varepsilon}\frac{2m V_{step}\psi}{\hbar^2}dx + \frac{2m}{\hbar^2}\frac{\hbar}{8}(\sqrt{3}-1)\sqrt{\frac{V_0}{m}}\int_{a-\varepsilon}^{a+\varepsilon}\delta(x-a)\psi\,dx - \frac{2mE}{\hbar^2}\int_{a-\varepsilon}^{a+\varepsilon}\psi\,dx$$

$$\lim_{\varepsilon\to 0}\left(\left.\frac{d\psi}{dx}\right|_{a+\varepsilon} - \left.\frac{d\psi}{dx}\right|_{a-\varepsilon} = \underbrace{\frac{2m}{\hbar^2}\int_{a-\varepsilon}^{a+\varepsilon}V_{step}\cdot\psi\,dx}_{\to 0} + \frac{m}{\hbar}(\sqrt{3}-1)\sqrt{\frac{V_0}{m}}\underbrace{\int_{a-\varepsilon}^{a+\varepsilon}\delta(x-a)\psi(x)\,dx}_{\psi(a)} - \underbrace{\frac{2mE}{\hbar^2}\int_{a-\varepsilon}^{a+\varepsilon}\psi(x)\,dx}_{\to 0}\right)$$

$$\boxed{\left.\frac{d\psi}{dx}\right|_{a^+} - \left.\frac{d\psi}{dx}\right|_{a^-} = \underline{(\sqrt{3}-1)\frac{\sqrt{V_0 m}}{\hbar}}\,\psi(a)}$$

$$\left.\frac{d\psi_3}{dx}\right|_{x=a} - \left.\frac{d\psi_2}{dx}\right|_{x=a} = (\sqrt{3}-1)\frac{\sqrt{V_0 m}}{\hbar}\underline{\psi_2(a)}$$

$V_0 = 2E$

$$K = \frac{\sqrt{2mE}}{\hbar}$$

$$g = \frac{\sqrt{2m(V_0-E)}}{\hbar} = \frac{\sqrt{2mE}}{\hbar} = K$$

$$-Cg e^{-ga} - AK\cos Ka = (\sqrt{3}-1)\frac{\sqrt{\overset{2E}{V_0}\, m}}{\hbar}A\sin Ka$$

$$\frac{\sqrt{2mE}}{\hbar} = K$$

$$A\sin Ka = C e^{-ga}$$

Continuity

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$$-\underbrace{C\cancel{K}e^{-Ka}}_{-A\sin Ka} - A\cancel{K}\cos Ka = (\sqrt{3}-1)\cancel{K}A\sin Ka$$

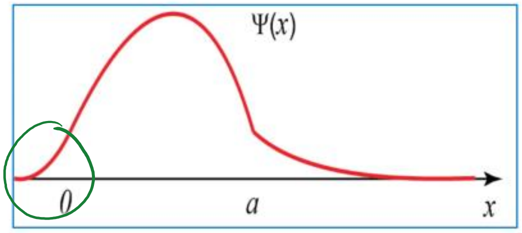
$$-A\cancel{\sin Ka} - \cancel{A}\cos Ka = \sqrt{3}\cancel{A}\sin KA - A\cancel{\sin Ka}$$

$$-\cos Ka = \sqrt{3}\sin Ka \Rightarrow -1 = \sqrt{3}\tan Ka$$

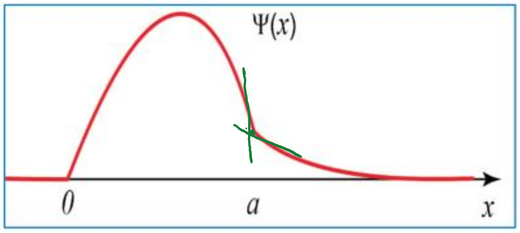
$$\tan Ka = -\frac{1}{\sqrt{3}} \Rightarrow Ka = \frac{5\pi}{6} + \pi n \Rightarrow K = \frac{5\pi}{6a} + \frac{\pi n}{a}$$

$$\frac{\sqrt{2mE}}{\hbar} = \frac{5\pi}{6a} + \cancel{\frac{\pi n}{a}} \qquad \frac{2mE}{\hbar^2} = \frac{25\pi^2}{36a^2} \qquad \boxed{E_g = \frac{25\pi^2\hbar^2}{72ma^2}}$$

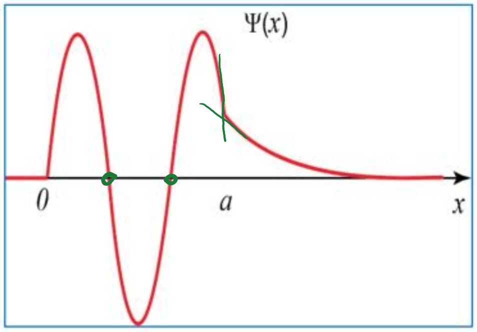
b) Which figure might be appropriate for the ground state wave function in part (a)?



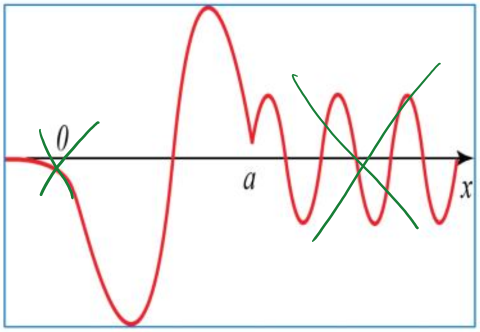
~~(a)~~



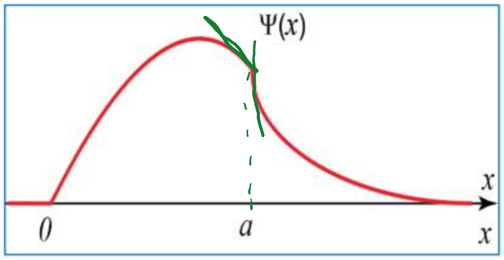
[✓ (b)]



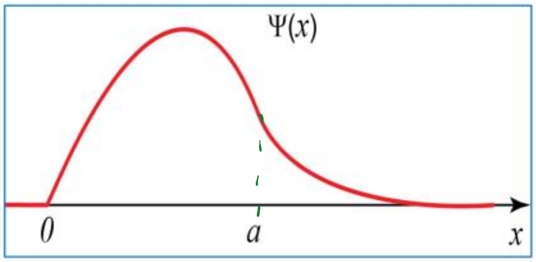
~~(c)~~



~~(d)~~



~~(e)~~



~~(f)~~

**Problem 2 (40 points).**

A particle of mass *m* in the one-dimensional harmonic oscillator at time *t*=*0* is in the state

$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$

$$|\psi(0)\rangle = \frac{1}{\sqrt{3}}\underset{n=1}{\underset{\uparrow}{|1\rangle}} + i\sqrt{\frac{2}{3}}\underset{n=2}{\underset{\uparrow}{|2\rangle}}$$

$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$

a) Determine the expectation value $\langle p_x\rangle$ of the momentum at time *t*.

$$\underline{|\psi(t)\rangle = \frac{1}{\sqrt{3}}e^{\frac{-i\cancel{\hbar}\omega(1+\frac{1}{2})t}{\cancel{\hbar}}}|1\rangle + i\sqrt{\frac{2}{3}}\,e^{\frac{-i\cancel{\hbar}\omega(2+\frac{1}{2})t}{\cancel{\hbar}}}|2\rangle} = \cancel{e^{-i\frac{3}{2}\omega t}}\left(\frac{1}{\sqrt{3}}|1\rangle + i\sqrt{\frac{2}{3}}\,e^{-i\omega t}|2\rangle\right)$$

$$\hat{p}_x|\psi(t)\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}-\hat{a}^+\right)\left(\frac{1}{\sqrt{3}}|1\rangle + i\sqrt{\frac{2}{3}}\,e^{-i\omega t}|2\rangle\right) =$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left(\frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}\,e^{-i\omega t}\sqrt{2}|1\rangle - \frac{1}{\sqrt{3}}\sqrt{2}|2\rangle - i\sqrt{\frac{2}{3}}\,e^{-i\omega t}\sqrt{3}|3\rangle\right)$$

$$\langle p_x\rangle = \langle\psi(t)|\hat{p}|\psi(t)\rangle = \left(\frac{1}{\sqrt{3}}\langle 1| - i\sqrt{\frac{2}{3}}\,e^{i\omega t}\langle 2|\right)\left(-i\sqrt{\frac{m\omega\hbar}{2}}\right)\left(\frac{1}{\sqrt{3}}|0\rangle + i\sqrt{\frac{2}{3}}\,e^{-i\omega t}\sqrt{2}|1\rangle - \right.$$

$$\left. -\frac{1}{\sqrt{3}}\sqrt{2}|2\rangle - i\sqrt{\frac{2}{3}}\,e^{-i\omega t}\sqrt{3}|3\rangle\right) = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\frac{1}{\sqrt{3}}\cdot i\sqrt{\frac{2}{3}}\,e^{-i\omega t}\sqrt{2} + i\sqrt{\frac{2}{3}}\,e^{i\omega t}\cdot\sqrt{\frac{2}{3}}\right)$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left(\frac{2}{3}i\,e^{-i\omega t} + \frac{2}{3}i\,e^{i\omega t}\right) = \sqrt{\frac{m\omega\hbar}{2}}\,\frac{2\cdot 2}{3}\underbrace{\frac{\left(e^{-i\omega t} + e^{i\omega t}\right)}{2}}_{\cos\omega t}$$

$$\boxed{\langle p_x\rangle = \frac{4}{3}\sqrt{\frac{m\omega\hbar}{2}}\cos\omega t}$$

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$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle = \int_{-\infty}^{\infty}\langle\psi|x\rangle\underbrace{\langle x|\hat{A}|\psi\rangle}\,dx$$

$\uparrow\ \hat{1} = \int |x\rangle\langle x|\,dx$

$$\boxed{\langle p_x\rangle} = \int_{-\infty}^{\infty}\underbrace{\langle\psi|x\rangle}_{\psi^*(x)}\underbrace{\langle x|p_x|\psi\rangle}_{\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle}dx = \int_{-\infty}^{\infty}\psi^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)\,dx$$

$$\sqrt{\frac{1}{3}}\psi_1(x) - i\sqrt{\frac{2}{3}}\psi_2(x) = \psi^*(x)$$

b) What is the physical meaning of the following integral?

$$\int_{-\infty}^{\infty}\overbrace{\left(\sqrt{\frac{1}{3}}\overbrace{\left(\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right)^{\frac{1}{4}}xe^{-\frac{m\omega x^2}{2\hbar}}}^{\psi_1(x)} - i\sqrt{\frac{2}{3}}\overbrace{\left(\frac{m\omega}{4\pi\hbar}\right)^{\frac{1}{4}}\left(2\frac{m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega x^2}{2\hbar}}}^{\psi_2(x)}\right)}^{\psi^*(x)}$$

$$\times\frac{\hbar}{i}\frac{\partial}{\partial x}\underbrace{\left(\sqrt{\frac{1}{3}}\underbrace{\left(\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right)^{\frac{1}{4}}xe^{-\frac{m\omega x^2}{2\hbar}}}_{\psi_1(x)} + i\sqrt{\frac{2}{3}}\underbrace{\left(\frac{m\omega}{4\pi\hbar}\right)^{\frac{1}{4}}\left(2\frac{m\omega}{\hbar}x^2-1\right)e^{-\frac{m\omega x^2}{2\hbar}}}_{\psi_2(x)}\right)}_{\psi(x)}dx =$$

$$= \int_{-\infty}^{\infty}\psi^*(x)\,\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)\,dx = \boxed{\langle p_x\rangle}$$

Physical meaning: $\langle\psi(0)|\,\hat{p}_x\,|\psi(0)\rangle$ ← this is the value for $\langle p_x\rangle$ at $t=0$.

c) What is the value of this integral? $\frac{4}{3}\sqrt{\frac{m\omega\hbar}{2}}\,\underbrace{\cos\omega 0}_{1} = \boxed{\frac{4}{3}\sqrt{\frac{m\omega\hbar}{2}}}$

6.19 , 7.7