Summary of Previous Lecture

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi_n(x)=E_n\psi_n(x)$$

$$\psi(x)=\langle x|E\rangle$$

$$\psi_n(x,t)=e^{-\frac{iE_nt}{\hbar}}\psi_n(x)$$

1) $\psi(x)$ is continuous

2) $$\left(\frac{d\psi}{dx}\right)_{x_0^+}-\left(\frac{d\psi}{dx}\right)_{x_0^-}=\frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon}V(x)\psi\,dx$$

If $V(x)$ is finite at x_0 $\Rightarrow$ $\left(\frac{d\psi}{dx}\right)_{x_0^+}=\left(\frac{d\psi}{dx}\right)_{x_0^-}$ $\Rightarrow$ $\frac{d\psi}{dx}$ is continuous at x_0

3) If the particle is (classically) constrained it's energy eigenvalues are discrete (quantized)

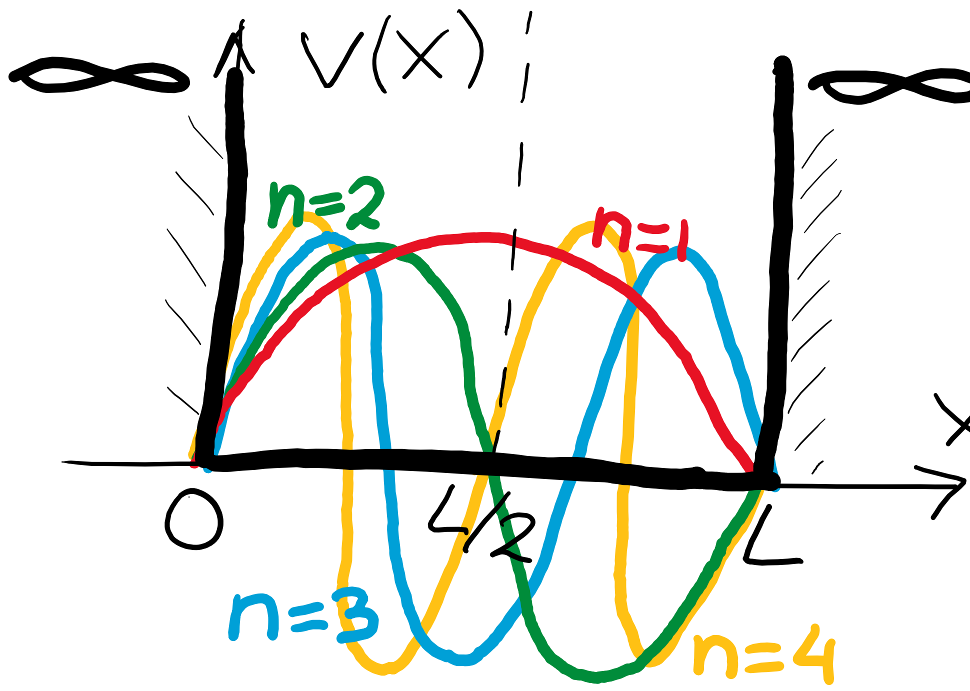
Particle in a Box (Infinite Quantum Well)

$$V(x) = \begin{cases} 0 & 0<x<L \\ \infty & \text{elsewhere} \end{cases}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\frac{\pi n x}{L} & 0<x<L \\ 0 & \text{elsewhere} \end{cases}$$

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

$n = 1, 2, 3, \ldots$



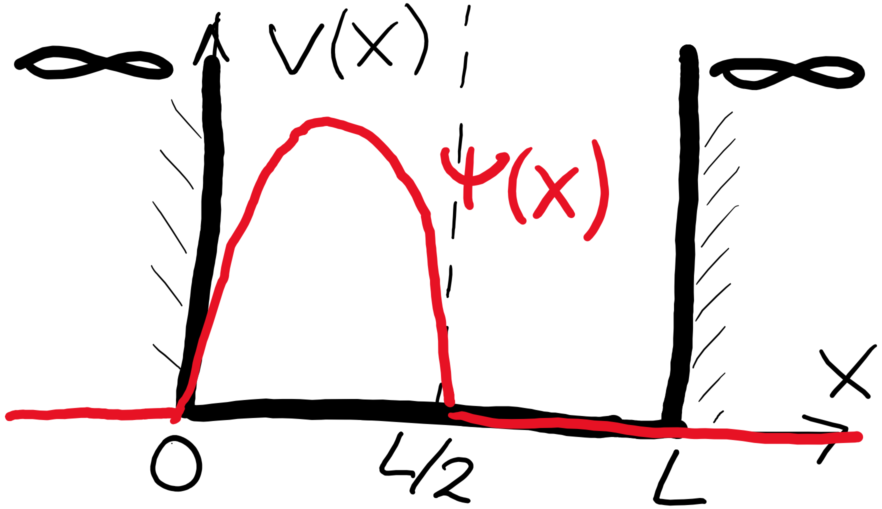
Can a particle have $\psi(x)$ that is not one of $\psi_n(x)$?

If $\psi(x)=0$ for $x<0$ and $x>L$

If $\psi(x)$ is continuous

If $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

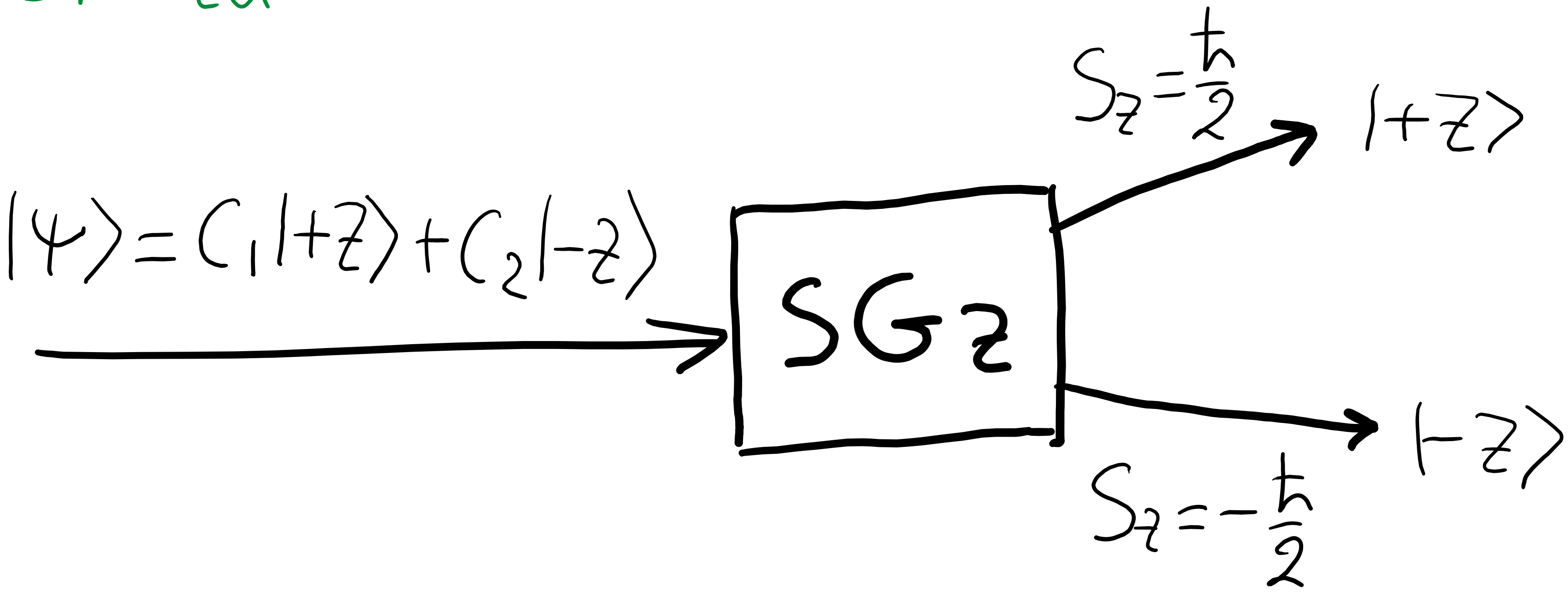
Yes!



$$\Psi(x) = \sum_n c_n \psi_n(x) \qquad c_n = \langle \psi_n | \Psi \rangle = \int_{-\infty}^{\infty} \psi_n^*(x) \Psi(x)\, dx$$

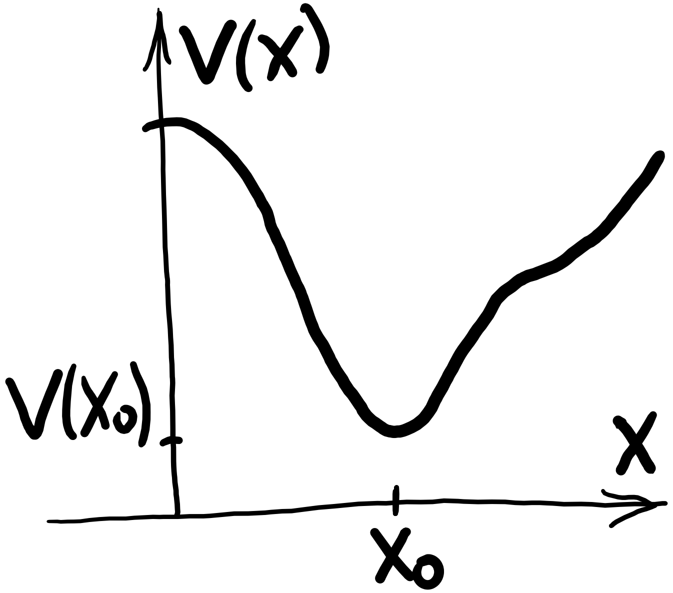
$$\Psi(x,t) = \sum_n c_n e^{-\frac{iE_n t}{\hbar}} \psi_n(x)$$

Similar to SG Experiment:



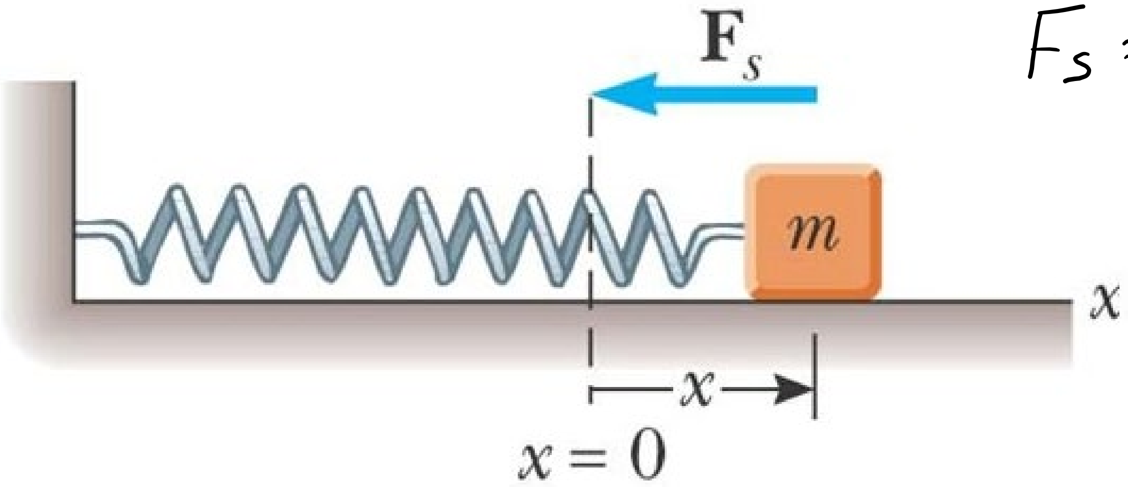
Harmonic Oscillator

$$V(x)=V(x_0)+\cancelto{0}{\left(\frac{dV}{dx}\right)_{x_0}}(x-x_0)+\frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x_0}(x-x_0)^2+\dots$$



$$V(x)=\frac{1}{2}kx^2$$

Classical oscillator



$$F_s=-kx=ma \Rightarrow a=\frac{d^2x}{dt^2}=-\overbrace{\frac{k}{m}}^{\omega^2}x$$

$$x(t)=A\cos(\omega t) \qquad k=m\omega^2$$

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

$$\left[\frac{p_x}{m\omega}\right] = \frac{\text{mass} \cdot (\text{distance}/\text{time})}{\text{mass} \cdot (1/\text{time})} = \text{distance}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p}_x \right)$$

$$\left[\sqrt{\frac{m\omega}{\hbar}}\, x\right] = \sqrt{\frac{\text{mass} \cdot (1/\text{time})}{\text{energy} \cdot \text{time}}} \cdot \text{distance}$$

$$\hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p}_x \right)$$

$$= \sqrt{\frac{1}{\text{distance}^2}} \cdot \text{distance} = 1$$

$$[\hat{a}, \hat{a}^{+}] = \frac{m\omega}{2\hbar} \left[\hat{x} + \frac{i}{m\omega} \hat{p}_x , \hat{x} - \frac{i}{m\omega} \hat{p}_x \right] = \frac{m\omega}{2\hbar} \left(\underbrace{[\hat{x}, \hat{x}]}_{0} - \frac{i}{m\omega} \underbrace{[\hat{x}, \hat{p}_x]}_{i\hbar} + \frac{i}{m\omega} \underbrace{[\hat{p}_x, \hat{x}]}_{-i\hbar} + \frac{1}{m\omega} \underbrace{[\hat{p}_x, \hat{p}_x]}_{0} \right)$$

$$[\hat{a}, \hat{a}^{+}] = \frac{m\omega}{2\hbar} \left(-\frac{i}{m\omega} \cdot i\hbar + \frac{i}{m\omega} (-i\hbar) \right) = 1$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)$$

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}_x\right)$$

$$\Rightarrow \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^+\right)$$

$$\hat{p}_x = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^+\right)$$

IMPORTANT

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{1}{2m}\cdot\frac{-m\omega\hbar}{2}\left(\hat{a} - \hat{a}^+\right)^2 + \frac{1}{2}m\omega^2\cdot\frac{\hbar}{2m\omega}\left(\hat{a} + \hat{a}^+\right)^2$$

$$= -\frac{\omega\hbar}{4}\left(\hat{a}^2 - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} + \hat{a}^{+2}\right) + \frac{\omega\hbar}{4}\left(\hat{a}^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^{+2}\right) = \frac{\omega\hbar}{4}\left(2\hat{a}\hat{a}^+ + 2\hat{a}^+\hat{a}\right)$$

$$\hat{H} = \frac{\omega\hbar}{2}\left(1 + \hat{a}^+\hat{a} + \hat{a}^+\hat{a}\right) = \omega\hbar\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

$$[\hat{a}, \hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$$

$$\hat{a}\hat{a}^+ = 1 + \hat{a}^+\hat{a}$$

$$\hat{H}=\hbar\omega\left(\hat{a}^{+}\hat{a}+\frac{1}{2}\right)=\hbar\omega\left(\hat{N}+\frac{1}{2}\right) \qquad \hat{N}=\hat{a}^{+}\hat{a}$$

$$[\hat{A}\hat{B},\hat{C}]=\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}$$

$$[\hat{N},\hat{a}]=[\underset{\hat{A}}{\hat{a}^{+}}\underset{\hat{B}}{\hat{a}},\underset{\hat{C}}{\hat{a}}]=\hat{a}^{+}\underbrace{[\hat{a},\hat{a}]}_{0}+\underbrace{[\hat{a}^{+},\hat{a}]}_{-1}\hat{a}=-\hat{a} \qquad \hat{N}\hat{a}=-\hat{a}+\hat{a}\hat{N}$$

$$[\hat{N},\hat{a}^{+}]=[\hat{a}^{+}\hat{a},\hat{a}^{+}]=\hat{a}^{+}\underbrace{[\hat{a},\hat{a}^{+}]}_{1}+\underbrace{[\hat{a}^{+},\hat{a}^{+}]}_{0}\hat{a}=\hat{a}^{+} \qquad \hat{N}\hat{a}^{+}=\hat{a}^{+}+\hat{a}^{+}\hat{N}$$

$\hat{N}$ is Hermitian, $[\hat{H},\hat{N}]=0$ $\qquad \hat{N}|\eta\rangle=\eta|\eta\rangle$

$$\hat{N}(\hat{a}|\eta\rangle)=(-\hat{a}+\hat{a}\hat{N})|\eta\rangle=(-\hat{a}+\hat{a}\eta)|\eta\rangle=(\eta-1)(\hat{a}|\eta\rangle)$$

$\hat{a}|\eta\rangle=C_{-}|\eta-1\rangle$ Lowering operator

$$\hat{N}(\hat{a}^{+}|\eta\rangle)=(\hat{a}^{+}+\hat{a}^{+}\hat{N})|\eta\rangle=(\hat{a}^{+}+\hat{a}^{+}\eta)|\eta\rangle=(\eta+1)(\hat{a}^{+}|\eta\rangle)$$

$\hat{a}^{+}|\eta\rangle=C_{+}|\eta+1\rangle$ raising operator

$$\hat{H} = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \qquad \hat{N} = \hat{a}^{+}\hat{a}$$

$$\hat{N}|\eta\rangle = \eta|\eta\rangle \Longrightarrow \hat{H}|\eta\rangle = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)|\eta\rangle = \hbar\omega\left(\eta + \frac{1}{2}\right)|\eta\rangle$$

There should be η_{min}

$$\hat{a}|\eta\rangle = C_{-}|\eta - 1\rangle \Longrightarrow \hat{a}|\eta_{min}\rangle = 0 \Longrightarrow \hat{a}^{+}\hat{a}|\eta_{min}\rangle = 0$$

$$\hat{N}|\eta_{min}\rangle = \eta_{min}|\eta_{min}\rangle = 0 \Longrightarrow \eta_{min} = 0$$

IMPORTANT

$$\hat{N}|n\rangle = n|n\rangle \qquad n = 0, 1, 2, \ldots$$

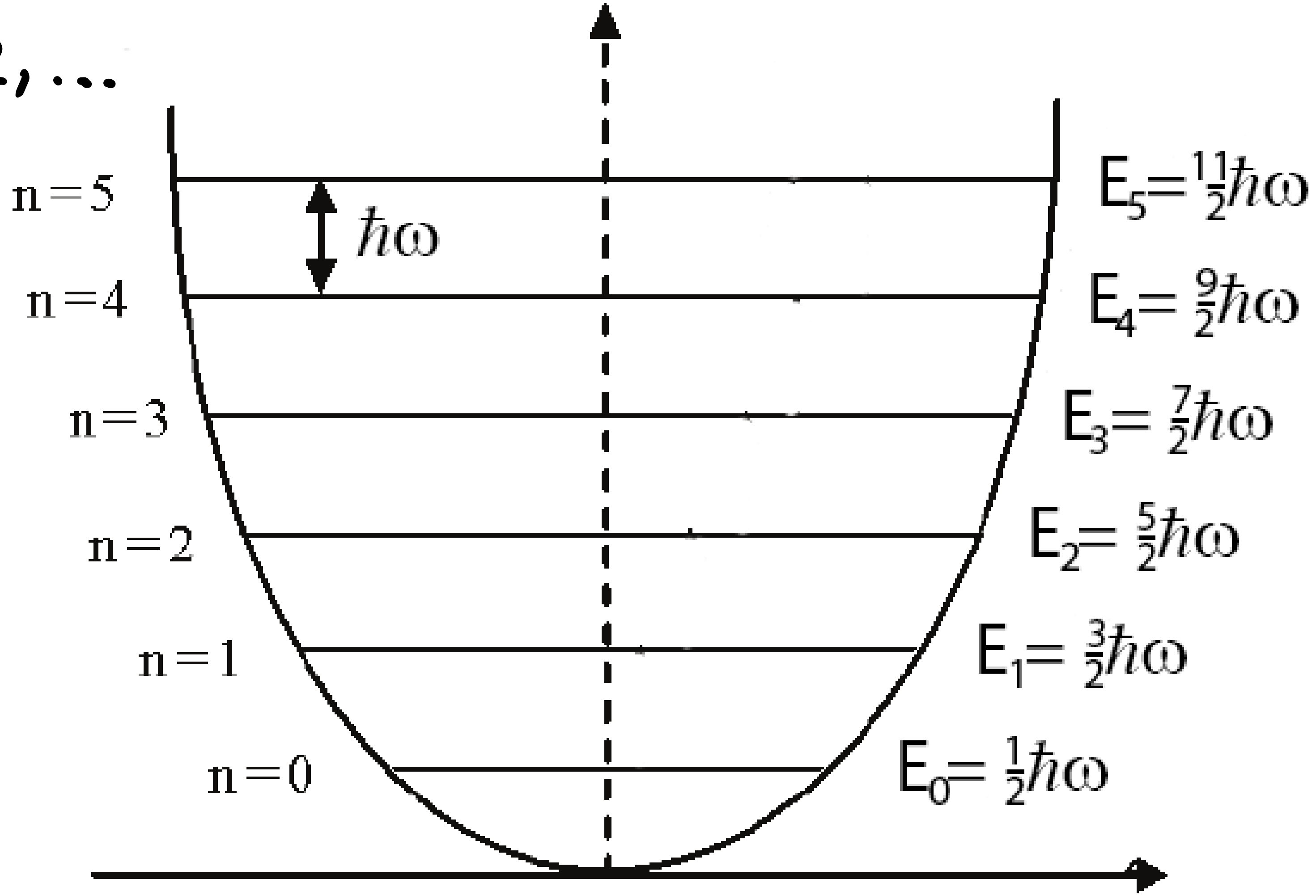
$$\hat{H}|n\rangle = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)|n\rangle = \underbrace{\hbar\omega\left(n + \frac{1}{2}\right)}_{E_n}|n\rangle$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) \qquad n = 0, 1, 2, \ldots$$

$$E_0 = \frac{\hbar\omega}{2} > 0$$

ground state energy



If $E_0 = 0$ then the particle is at rest at origin $\Rightarrow$ $\Delta x = 0$ $\Delta p_x = 0$

But $\Delta x \Delta p_x \geq \frac{\hbar}{2}$!

$$[\hat{a},\hat{a}^+]=\hat{a}\hat{a}^+-\hat{a}^+\hat{a}=1$$

$$\hat{a}\hat{a}^+=1+\hat{a}^+\hat{a}=1+\hat{N}$$

$$\hat{a}^+|n\rangle = C_+|n+1\rangle$$

$$\langle n|\hat{a} = C_+^*\langle n+1|$$

$$\langle n|\hat{a}\hat{a}^+|n\rangle = C_+C_+^*\underbrace{\langle n+1|n+1\rangle}_{1}$$

$$\langle n|1+\hat{N}|n\rangle = |C_+|^2$$

$$1+n=|C_+|^2 \Rightarrow C_+=\sqrt{1+n} \Rightarrow \hat{a}^+|n\rangle=\sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}|n\rangle = C_-|n-1\rangle$$

$$\langle n|\hat{a}^+ = C_-^*\langle n-1|$$

$$\langle n|\hat{a}^+\hat{a}|n\rangle = C_-^*C_-\langle n-1|n-1\rangle$$

IMPORTANT

$$\langle n|\hat{N}|n\rangle=|C_-|^2 \Rightarrow n=|C_-|^2 \Rightarrow C_-=\sqrt{n} \Rightarrow \hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle$$

$$\hat{a}^+|0\rangle=\sqrt{1+0}\,|1\rangle \qquad |1\rangle=\hat{a}^+|0\rangle$$

$$\hat{a}^+|1\rangle=\sqrt{1+1}\,|2\rangle \qquad |2\rangle=\frac{1}{\sqrt{2}}\hat{a}^+|1\rangle=\frac{1}{\sqrt{2}}\left(\hat{a}^+\right)^2|0\rangle$$

$$\hat{a}^+|2\rangle=\sqrt{2+1}\,|3\rangle \qquad |3\rangle=\frac{1}{\sqrt{3}}\hat{a}^+|2\rangle=\frac{1}{\sqrt{2\cdot3}}\left(\hat{a}^+\right)^3|0\rangle$$

$$\hat{a}^+|3\rangle=\sqrt{3+1}\,|4\rangle \qquad |4\rangle=\frac{1}{\sqrt{4}}\hat{a}^+|3\rangle=\frac{1}{\sqrt{2\cdot3\cdot4}}\left(\hat{a}^+\right)^4|0\rangle$$

$$|n\rangle=\frac{1}{\sqrt{n!}}\left(\hat{a}^+\right)^n|0\rangle$$