Summary of Previous Lecture

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x)$$

$$\Psi(x,t)=e^{-\frac{iEt}{\hbar}}\psi(x)$$

1) $\psi(x)$ is continuous

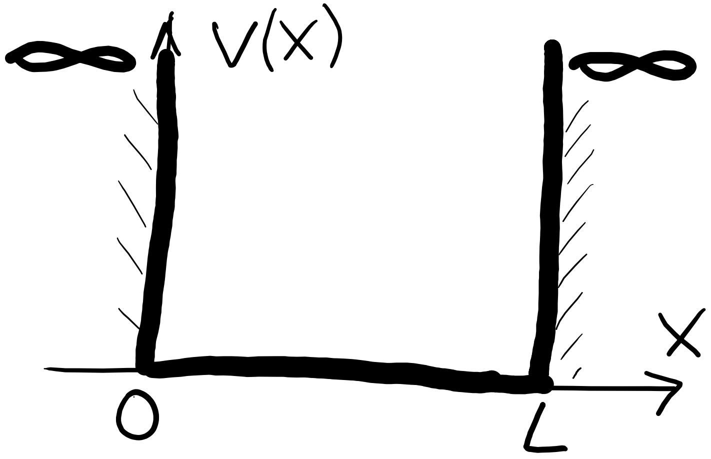
2) $$\left(\frac{d\psi}{dx}\right)_{x_0^+}-\left(\frac{d\psi}{dx}\right)_{x_0^-}=\frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon}V(x)\psi\,dx$$

If $V(x)$ is finite at x_0 $\Rightarrow$ $\left(\frac{d\psi}{dx}\right)_{x_0^+}=\left(\frac{d\psi}{dx}\right)_{x_0^-}$ $\Rightarrow$ $\frac{d\psi}{dx}$ is continuous at x_0

3) If the particle is (classically) constrained it's energy eigenvalues are discrete (quantized)

Particle in a Box (Infinite Quantum Well)

$$V(x) = \begin{cases} 0 & 0<x<L \\ \infty & \text{elsewhere} \end{cases}$$



$$\hat{1} = \int dx\, |x\rangle\langle x|$$

$$\langle V\rangle = \langle\psi|\hat{V}|\psi\rangle = \int_{-\infty}^{\infty} \langle\psi|V(x)|x\rangle\langle x|\psi\rangle\, dx = \int_{-\infty}^{\infty} V(x)|\psi(x)|^2 dx$$

Take $a<b<0$ $\quad V=\infty$ for $a<x<b$

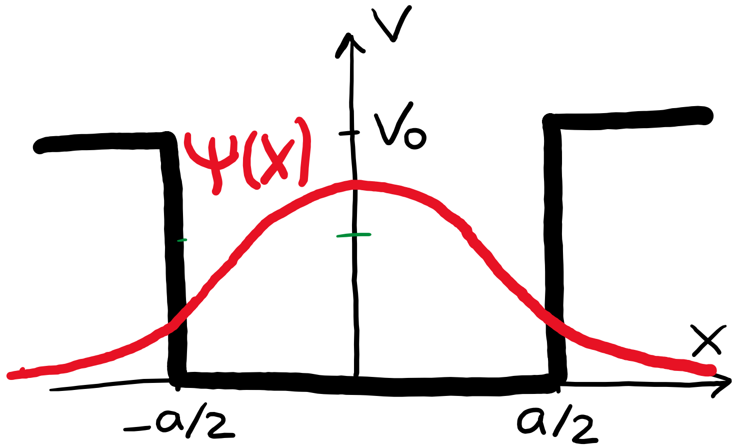
$$\int_a^b V(x)|\psi(x)|^2 dx = \infty \quad \text{if} \quad \psi(x) \neq 0 \quad \text{for} \quad a<x<b$$

$$\psi(x) = 0 \quad \text{for} \quad x<0,\ x>L$$

Recall finite Well:

For $x < -a/2$: $\psi(x) = C e^{gx}$

For $x > a/2$: $\psi(x) = D e^{-gx}$



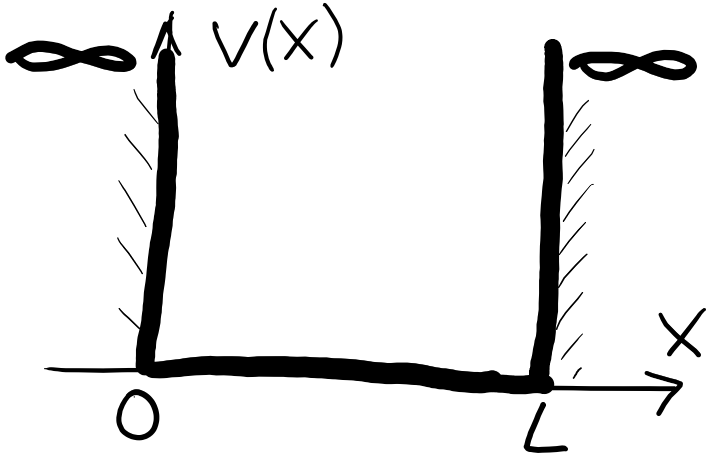
$$g = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

as $V_0 \to \infty$ $\quad g \to \infty \quad \psi(x) \to 0$

in the "forbidden" region
$x < -a/2$ and $x > a/2$

$0 < x < L$

$$V(x) = \begin{cases} 0 & 0<x<L \\ \infty & \text{elsewhere} \end{cases}$$



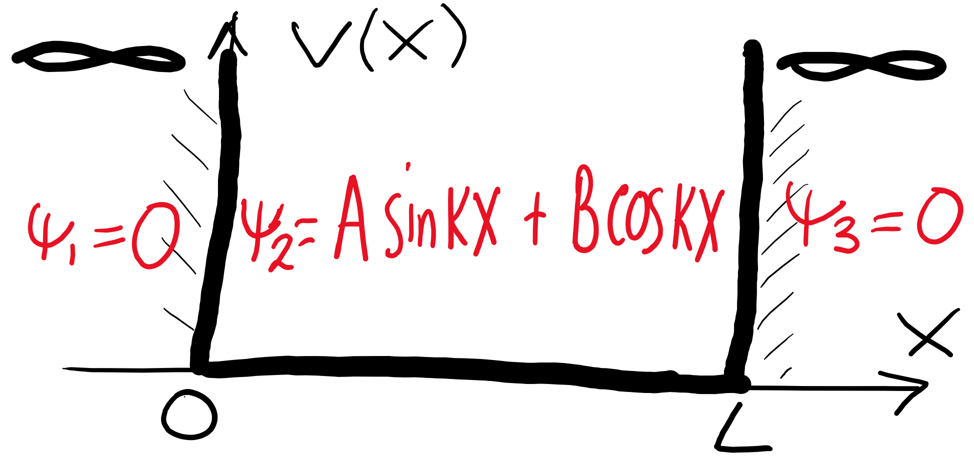
$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \Rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi$$

$$\frac{2mE}{\hbar^2} \rightarrow k^2$$

$\psi(x) = A\sin kx + B\cos kx$ or $\psi(x) = A' e^{ikx} + B' e^{-ikx}$

Matching



1) $x=0$

Continuity: $\psi_1\big|_{x=0}=\psi_2\big|_{x=0}$

$0=A\sin k\cdot 0+B\cos k\cdot 0 \Rightarrow 0=B \Rightarrow \psi_2=A\sin kx$

Smoothness? $V=\infty$ at $x=0 \Rightarrow$ NO!

2) $x=L$

Continuity: $\psi_2\big|_{x=L}=\psi_3\big|_{x=L} \Rightarrow A\sin kL=0$

$kL=\pi n \Rightarrow k=\frac{\pi n}{L}=\sqrt{\frac{2mE}{\hbar^2}} \Rightarrow$

$n=1,2,3,\ldots$

$$\boxed{E_n=\frac{\hbar^2\pi^2n^2}{2mL^2}}$$

Smoothness? NO!

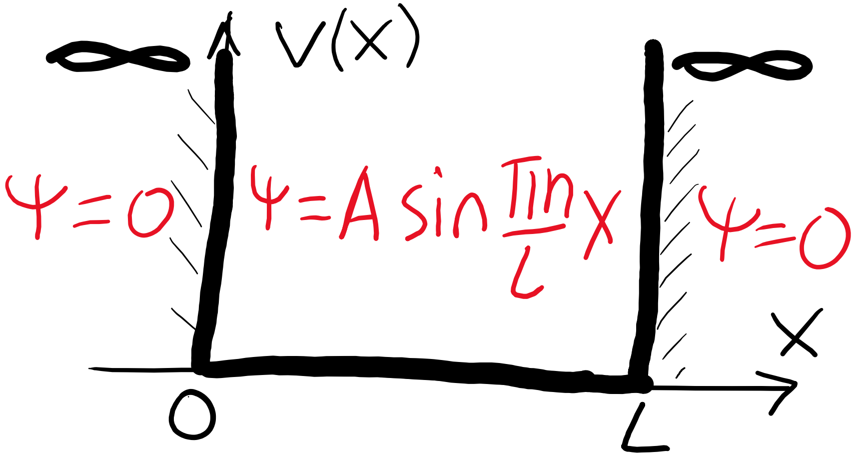
$E_4=\frac{\hbar^2\pi^2 4^2}{2mL^2}$

$E_3=\frac{\hbar^2\pi^2 3^2}{2mL^2}$

$E_2=\frac{\hbar^2\pi^2 2^2}{2mL^2}$

$E_1=\frac{\hbar^2\pi^2 1}{2mL^2}$

Normalization



$$\int_{-\infty}^{\infty}|\psi(x)|^2dx=1$$

$$\sin^2\alpha=\frac{1}{2}(1-\cos2\alpha)$$

$$1=|A|^2\int_0^L\sin^2\left(\frac{\pi n}{L}x\right)dx=|A|^2\cdot\frac{1}{2}\int_0^L\left(1-\cos\left(\frac{2\pi n}{L}x\right)\right)dx$$

$$=|A|^2\cdot\frac{1}{2}\left[x\Big|_0^L-\frac{L}{2\pi n}\sin\left(\frac{2\pi n}{L}x\right)\Big|_0^L\right]$$

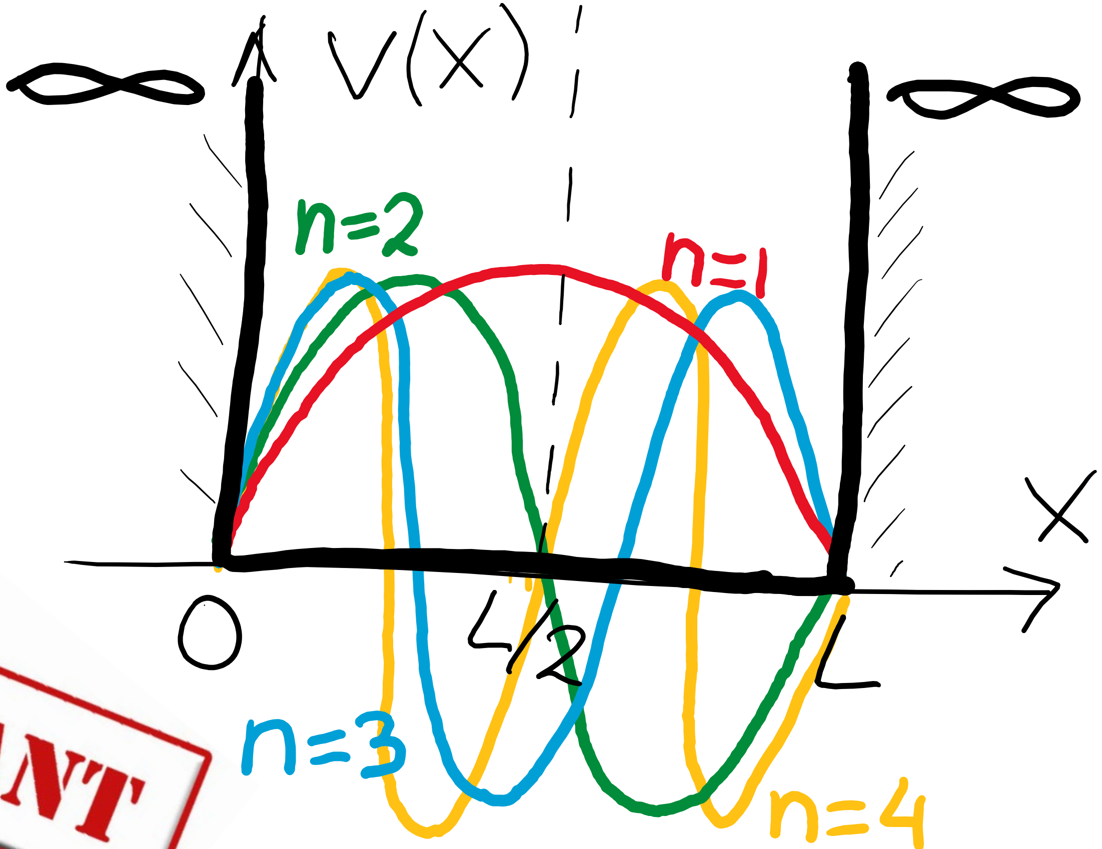
$$=|A|^2\frac{L}{2}=1 \implies A=\sqrt{\frac{2}{L}}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{\pi n x}{L} & 0<x<L \\ 0 & \text{elsewhere} \end{cases}$$

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2 m L^2} \qquad n = 1, 2, 3, \ldots$$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

IMPORTANT



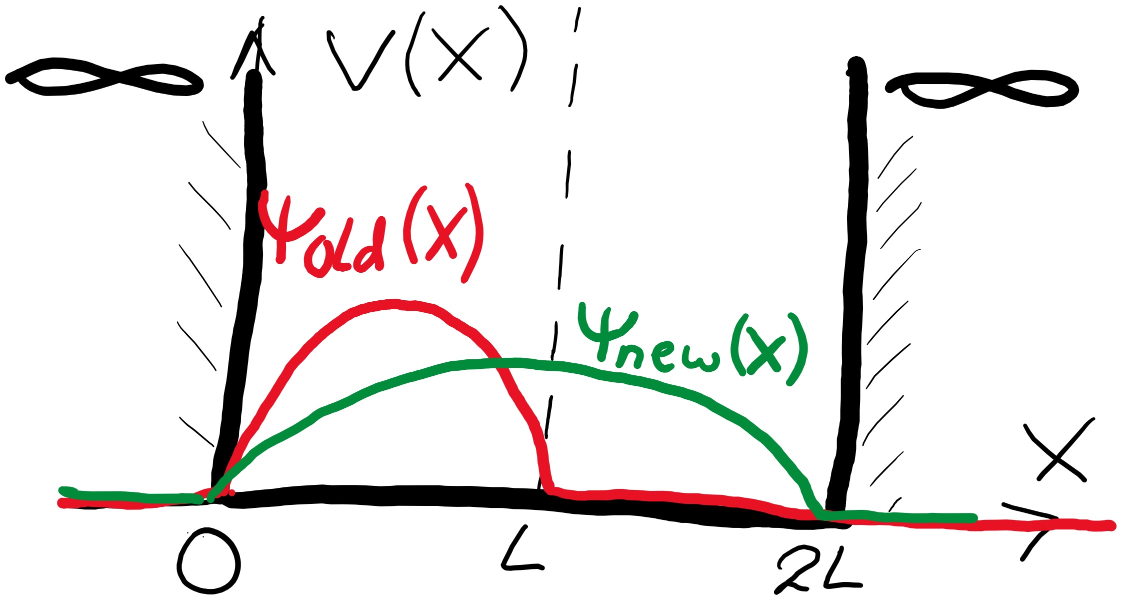
Symmetry: If $V(x)$ is symmetric $\Rightarrow$ $\psi(x)$ has symmetry (odd, even)

# of nodes: $\psi_{n+1}(x)$ has 1 more node compared with $\psi_n(x)$

Example 1 Infinite Quant. Well

Particle is in ground state in the infinite quantum well of size L

The $x=L$ boundary of the well is very rapidly moved to $x=2L$



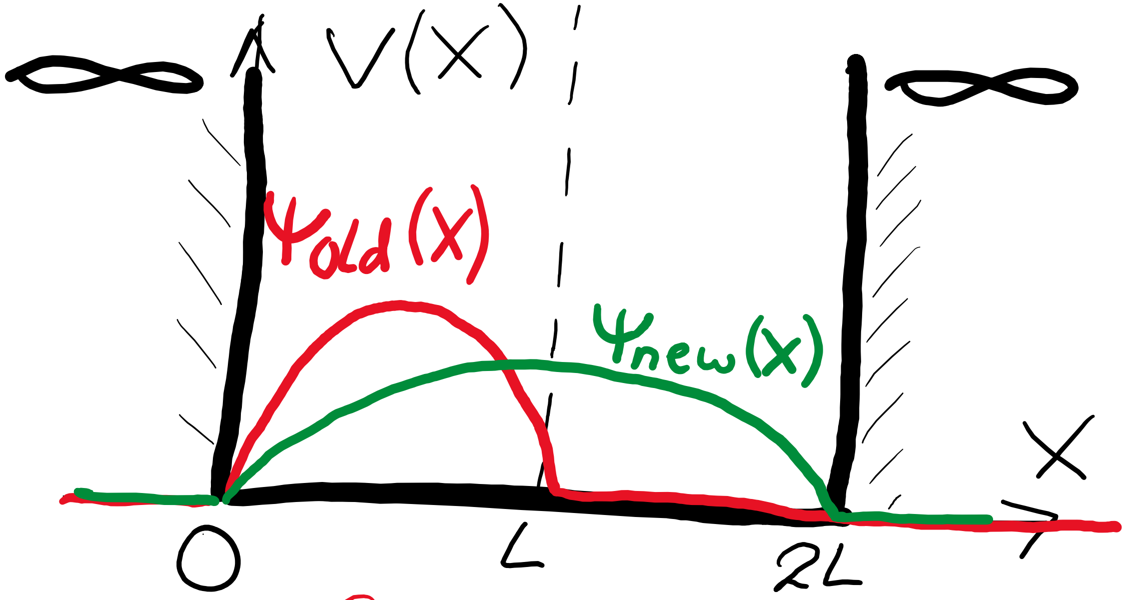
What is the probability the subsequent measurement of energy of the particle will yield the ground-state energy of the new well?

$$\mathbb{1} = \int dx\, |x\rangle\langle x|$$

$$P = |\langle \psi_{new} | \psi_{old} \rangle|^2$$

$$\langle \psi_{new} | \psi_{old} \rangle = \int \langle \psi_{new} | x \rangle \langle x | \psi_{old} \rangle dx = \int \psi^*_{new}(x)\, \psi_{old}(x)\, dx$$

$$\psi_{old}(x)=\begin{cases}\sqrt{\frac{2}{L}}\sin\frac{\pi}{L}x & 0<x<L\\ 0 & \text{elsewhere}\end{cases}$$



$$\psi_{new}(x)=\begin{cases}\sqrt{\frac{2}{2L}}\sin\frac{\pi x}{2L} & 0<x<2L\\ 0 & \text{elsewhere}\end{cases}$$

$$\langle\psi_{new}|\psi_{old}\rangle=\int_{-\infty}^{\infty}\psi^*_{new}(x)\psi_{old}(x)dx=\int_0^L\sqrt{\frac{2}{2L}}\sin\underbrace{\frac{\pi x}{2L}}_{\beta}\cdot\sqrt{\frac{2}{L}}\sin\underbrace{\frac{\pi x}{L}}_{\alpha}dx$$

$$\sin\alpha\sin\beta=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)$$

$$=\sqrt{\frac{1}{L}}\cdot\sqrt{\frac{2}{L}}\cdot\frac{1}{2}\int_0^L\left(\cos\left(\frac{\pi x}{L}-\frac{\pi x}{2L}\right)-\cos\left(\frac{\pi x}{L}+\frac{\pi x}{2L}\right)\right)dx=\frac{1}{L\sqrt{2}}\int_0^L\left(\cos\left(\frac{\pi x}{2L}\right)-\cos\left(\frac{3\pi x}{2L}\right)\right)dx$$

$$\langle \psi_{new} | \psi_{old} \rangle = \frac{1}{L\sqrt{2}} \int_0^L \left(\cos\left(\frac{\pi x}{2L}\right) - \cos\left(\frac{3\pi x}{2L}\right) \right) dx = \frac{1}{L\sqrt{2}} \left(\frac{2L}{\pi x} \sin\left(\frac{\pi x}{2L}\right) - \frac{2L}{3\pi} \sin\left(\frac{3\pi x}{2L}\right) \right) \Bigg|_0^L$$

$$= \frac{1}{L\sqrt{2}} \left(\frac{2L}{\pi} \sin\frac{\pi}{2} - \frac{2L}{3\pi} \sin\frac{3\pi}{2} \right) = \frac{1}{L\sqrt{2}} \left(\frac{2L}{\pi} + \frac{2L}{3\pi} \right) = \frac{4\sqrt{2}}{3\pi}$$

$$P = |\langle \psi_{new} | \psi_{old} \rangle|^2 = \left(\frac{4\sqrt{2}}{3\pi} \right)^2 = \frac{32}{9\pi^2}$$

✓

Example 2

Infinite Quant. Well

$$\psi(x) = \begin{cases} \frac{\sqrt{30}}{L^{5/2}} x(L-x) & 0<x<L \\ 0 & \text{elsewhere} \end{cases}$$

Energy is measured

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \qquad n = 1, 2, 3, \ldots$$

$$P\left(E = \frac{3\hbar^2\pi^2}{2mL^2}\right) = 0$$

$$P\left(E = \frac{9\hbar^2\pi^2}{2mL^2}\right) = |\langle \psi_3 | \psi \rangle|^2 = \left(\frac{8}{9\pi^3}\sqrt{\frac{5}{3}}\right)^2 = \frac{320}{243\pi^6}$$

$$\langle \psi_3 | \psi \rangle = \int_{-\infty}^{\infty} \psi_3^*(x)\psi(x)\,dx = \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi x}{L}\right) \frac{\sqrt{30}}{L^{5/2}} x(L-x)\,dx = \frac{8}{9\pi^3}\sqrt{\frac{5}{3}}$$

Example 3

Infinite Quant. Well

$$\Psi(x,0)=\begin{cases}\frac{4}{\sqrt{5L}}\sin^3\left(\frac{\pi x}{L}\right) & 0<x<L\\ 0 & \text{elsewhere}\end{cases}$$

1) $\langle E\rangle$ – ?

2) $\Psi(x,t)$ – ?

Eigenstates of $\hat{H}$:

$$\psi_n(x)=\begin{cases}\sqrt{\frac{2}{L}}\sin\frac{\pi n x}{L} & 0<x<L\\ 0 & \text{elsewhere}\end{cases}$$

$$E_n=\frac{\hbar^2\pi^2 n^2}{2mL^2} \qquad n=1,2,3,\dots$$

$$\sin^3\alpha=\frac{3}{4}\sin\alpha-\frac{1}{4}\sin 3\alpha$$

$$\frac{4}{\sqrt{5L}}\sin^3\left(\frac{\pi x}{L}\right)=\frac{4}{\sqrt{5L}}\left(\frac{3}{4}\sin\left(\frac{\pi x}{L}\right)-\frac{1}{4}\sin\left(\frac{3\pi x}{L}\right)\right)=\frac{3}{\sqrt{5L}}\sin\left(\frac{\pi x}{L}\right)-\frac{1}{\sqrt{5L}}\sin\left(\frac{3\pi x}{L}\right)$$

$$=\frac{3}{\sqrt{10}}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)-\frac{1}{\sqrt{10}}\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right)=\frac{3}{\sqrt{10}}\psi_1(x)-\frac{1}{\sqrt{10}}\psi_3(x)$$

$$\Psi(x,0)=\frac{3}{\sqrt{10}}\psi_1(x)-\frac{1}{\sqrt{10}}\psi_3(x) \qquad E_n=\frac{\hbar^2\pi^2 n^2}{2mL^2}$$

$$\langle E\rangle=\left|\frac{3}{\sqrt{10}}\right|^2 E_1+\left|\frac{1}{\sqrt{10}}\right|^2 E_3=\frac{9}{10}\,\frac{\hbar^2\pi^2\cdot 1^2}{2mL^2}+\frac{1}{10}\,\frac{\hbar^2\pi^2 3^2}{2mL^2}=\frac{9\hbar^2\pi^2}{10mL^2}$$

$$\Psi(x,t)=\frac{3}{\sqrt{10}}e^{-\frac{iE_1 t}{\hbar}}\psi_1(x)-\frac{1}{\sqrt{10}}e^{-\frac{iE_3 t}{\hbar}}\psi_3(x)$$

$$=\frac{3}{\sqrt{10}}e^{-\frac{i\hbar\pi^2 t}{2mL^2}}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)-\frac{1}{\sqrt{10}}e^{-\frac{9i\hbar\pi^2 t}{2mL^2}}\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right)$$

Time Evolutio

Infinite Quant. Well

$$\Psi(x,0) = \begin{cases} \frac{4}{\sqrt{5L}} \sin^3\left(\frac{\pi x}{L}\right) & 0<x<L \\ 0 & \text{elsewher} \end{cases}$$

Time Evolutio

Infinite Quant. Well

$$\psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) & 0 < x < L \\ 0 & \text{elsewhere} \end{cases}$$

Time Evolution

Infinite Quant. Well

$$\Psi(x,0) = \begin{cases} \sqrt{\frac{1}{L}}\sin\left(\frac{\pi x}{L}\right) + \sqrt{\frac{1}{L}}\sin\left(\frac{2\pi x}{L}\right) & 0 < x < \\ 0 & \text{elsewhe} \end{cases}$$

Time Evolution

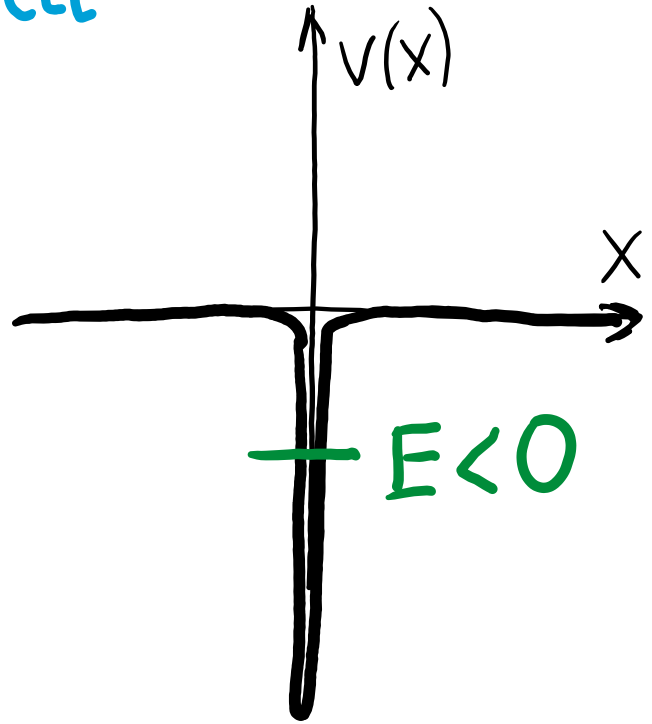
Infinite Quant. Well

$$\Psi(x,0)=\begin{cases}\sqrt{\dfrac{2}{L/2}}\sin\left(\dfrac{\pi x}{L/2}\right) & 0<x<\dfrac{L}{2}\\ 0 & \text{elsewhere}\end{cases}$$

Example 4

Not Infinite Quant. Well

$$V(x) = -\frac{\lambda}{b}\cdot\frac{\hbar^2}{2m}\,\delta(x)$$



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

For $x \neq 0$ $\quad V(x) = 0$:

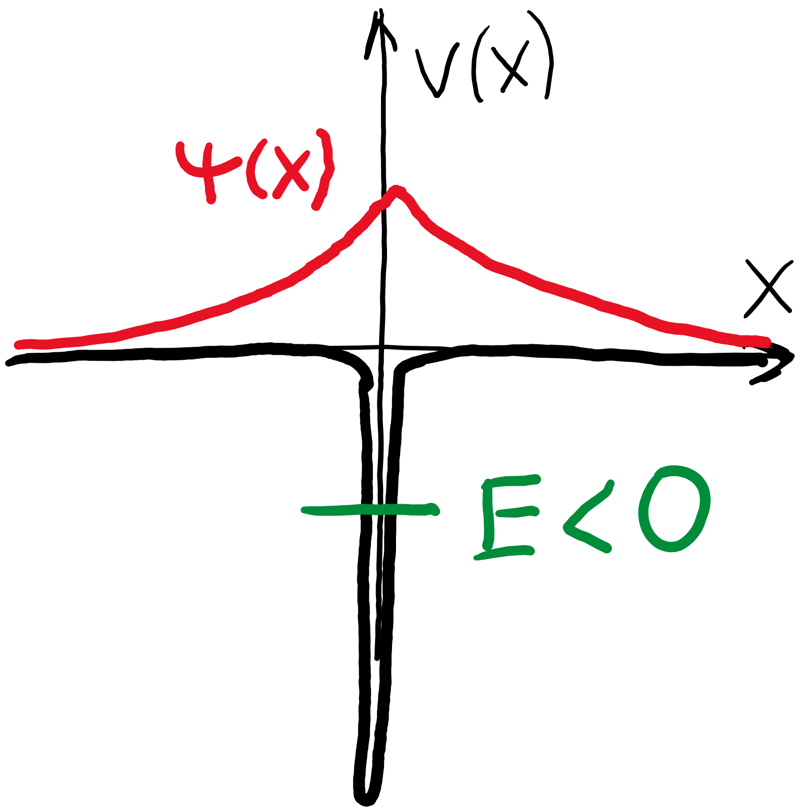
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \;\Rightarrow\; \frac{d^2\psi}{dx^2} = \underbrace{-\frac{2mE}{\hbar^2}}_{g^2}\psi \;\Rightarrow\; \frac{d^2\psi}{dx^2} = g^2\psi$$

$$\psi(x) = Ae^{gx} + Be^{-gx}$$

$$\psi(x) = A e^{qx} + B e^{-qx}$$

$x>0 \quad A=0 \quad \psi_1(x) = B e^{-qx}$

$x<0 \quad B=0 \quad \psi_2(x) = A e^{qx}$



$$\int_{-\varepsilon}^{\varepsilon} \frac{d^2\psi}{dx^2}\,dx = \int_{-\varepsilon}^{\varepsilon} \frac{2m}{\hbar^2}\left[V(x) - E\right]\psi\,dx$$

$$\lim_{\varepsilon \to 0} \left(\frac{d\psi}{dx}\right)_{\varepsilon} - \left(\frac{d\psi}{dx}\right)_{-\varepsilon} = \frac{2m}{\hbar^2}\int_{-\varepsilon}^{\varepsilon} V(x)\psi\,dx - \underbrace{\frac{2mE}{\hbar^2}\int_{-\varepsilon}^{\varepsilon} \psi\,dx}_{0}$$

$$\left(\frac{d\psi}{dx}\right)_{0^+} - \left(\frac{d\psi}{dx}\right)_{0^-} = \frac{2m}{\hbar^2} \lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} V(x)\psi\, dx = -\frac{\lambda}{\beta} \lim_{\varepsilon\to 0} \underbrace{\int_{-\varepsilon}^{\varepsilon} \delta(x)\psi\, dx}_{\psi(0)} = -\frac{\lambda}{\beta}\psi(0)$$

$$V(x) = -\frac{\lambda}{\beta}\cdot\frac{\hbar^2}{2m}\delta(x)$$

$x>0 \qquad \psi_1(x) = Be^{-qx}$

$x<0 \qquad \psi_2(x) = Ae^{qx}$

Matching at $x=0$:

1) Continuity: $\psi_1(0) = \psi_2(0) \Rightarrow Be^{-0} = Ae^{0} \Rightarrow A = B$

2) $$\left(\frac{d\psi_1}{dx}\right)_0 - \left(\frac{d\psi_2}{dx}\right)_0 = -\frac{\lambda}{\beta}\psi_1(0) \Rightarrow -Bqe^{0} - Aqe^{0} = -\frac{\lambda}{\beta}Be^{0} \Rightarrow -(B+A)q = -\frac{\lambda}{\beta}B$$

$$A = B$$

$$-(B+A)q = -\frac{\lambda}{\beta} B$$

$$\Bigg\} \quad -(A+A)q = -\frac{\lambda}{\beta} A \implies q = \frac{\lambda}{2\beta}$$

$$q^2 = -\frac{2mE}{\hbar^2} \implies -\frac{2mE}{\hbar^2} = \left(\frac{\lambda}{2\beta}\right)^2$$

$$E = -\frac{\lambda^2 \hbar^2}{8m\beta^2}$$