Short Summary

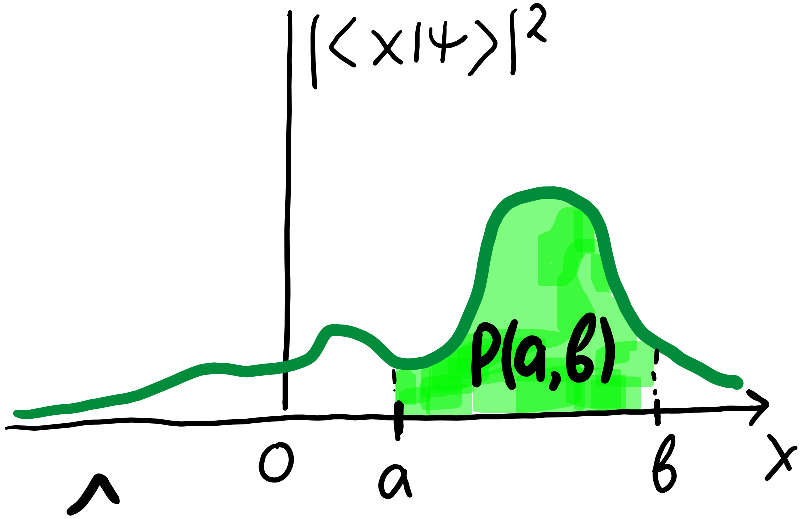
$$\hat{X}|x\rangle = x|x\rangle \qquad |\psi\rangle = \int_{-\infty}^{\infty} \underbrace{|x\rangle\langle x|}_{\hat{1}}\psi\rangle\, dx \qquad \hat{1} = \int_{-\infty}^{\infty} |x\rangle\langle x|\, dx$$

$$\langle x|x'\rangle = \delta(x-x')$$

Wave function

$$\langle x|\psi\rangle = \psi(x) \qquad \langle\psi|x\rangle = \psi^*(x)$$

$$P(a,b) = \int_a^b |\langle x|\psi\rangle|^2 dx = \int_a^b |\psi(x)|^2 dx$$



Translation Operator

$$\hat{T}(a)|x\rangle = |x+a\rangle \qquad \hat{T}^{\dagger}(a)\hat{T}(a) = \hat{1}$$

$$\hat{T}(dx)|x\rangle = |x+dx\rangle$$

$$\hat{T}(dx) = \hat{1} - \frac{i}{\hbar}\hat{p}_x dx$$

$$\hat{T}(a) = \lim_{N\to\infty}\left[1 - \frac{i}{\hbar}\hat{p}_x \frac{a}{N}\right]^N = e^{-\frac{i\hat{p}_x a}{\hbar}}$$

$$\hat{p}_x^\dagger = \hat{p}_x$$

$$[\hat{x}, \hat{T}(\delta x)] = \hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x} = \hat{x}\left(1 - \frac{i}{\hbar}\hat{p}_x \delta x\right) - \left(1 - \frac{i}{\hbar}\hat{p}_x \delta x\right)\hat{x}$$

$$= \left(\frac{-i \cdot \delta x}{\hbar}\right)(\hat{x}\hat{p}_x - \hat{p}_x \hat{x}) = \left(\frac{-i\,\delta x}{\hbar}\right)[\hat{x}, \hat{p}_x]$$

$$\left(\hat{x}\hat{T}(\delta x)-\hat{T}(\delta x)\hat{x}\right)|\psi\rangle=\left(-\frac{i\,\delta x}{\hbar}\right)[\hat{x},\hat{p}_x]|\psi\rangle$$

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$$\left(\hat{x}\hat{T}(\delta x)-\hat{T}(\delta x)\hat{x}\right)\int|x\rangle\langle x|\psi\rangle dx=\hat{x}\int|x+\delta x\rangle\langle x|\psi\rangle dx-\hat{T}(\delta x)\int x|x\rangle\langle x|\psi\rangle dx$$

$$=\int(x+\delta x)|x+\delta x\rangle\langle x|\psi\rangle dx-\int x|x+\delta x\rangle\langle x|\psi\rangle dx=\delta x\int|x+\delta x\rangle\langle x|\psi\rangle dx$$

$$\simeq\delta x\int|x\rangle\langle x|\psi\rangle dx=\delta x|\psi\rangle$$

$$\left(-\frac{i\,\delta x}{\hbar}\right)[\hat{x},\hat{p}_x]|\psi\rangle=\delta x|\psi\rangle \qquad -\frac{i\,\delta x}{\hbar}[\hat{x},\hat{p}_x]=\delta x$$

$$[\hat{x},\hat{p}_x]=i\hbar$$

Kinetic energy: $\frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})$$

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle + \langle\psi|\frac{\partial\hat{A}}{\partial t}|\psi\rangle$$

$$\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{x}]|\psi\rangle = \frac{i}{\hbar}\langle\psi|[\frac{\hat{p}_x^2}{2m},\hat{x}]|\psi\rangle + \frac{i}{\hbar}\langle\psi|[V(\hat{x}),\hat{x}]|\psi\rangle$$

$$= \frac{i}{2m\hbar}\langle\psi|[\hat{p}_x^2,\hat{x}]|\psi\rangle = \frac{i}{2m\hbar}\langle\psi|\left(\hat{p}_x[\hat{p}_x,\hat{x}] + [\hat{p}_x,\hat{x}]\hat{p}_x\right)|\psi\rangle = \frac{\langle\psi|\hat{p}_x|\psi\rangle}{m} = \frac{\langle p_x\rangle}{m}$$

$$\frac{d\langle p_x\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{p}_x]|\psi\rangle$$

$$[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$$

$$= -\langle\psi|\frac{d}{dx}V(\hat{x})|\psi\rangle = \left\langle -\frac{dV}{dx}\right\rangle$$

$$ma = \frac{dP}{dt} = F = -\frac{dV}{dx}$$

$$[\hat{x},\hat{p}_x]=i\hbar \qquad [\hat{A},\hat{B}]=i\hat{C} \Rightarrow \Delta A\cdot\Delta B \geq \frac{|\langle C\rangle|}{2} \qquad \Delta x\cdot\Delta p_x \geq \frac{\hbar}{2}$$

$$\hat{T}(\delta x)|\psi\rangle = \hat{T}(\delta x)\int |x\rangle\langle x|\psi\rangle dx = \int |x+\delta x\rangle\langle x|\psi\rangle dx = \int |x'\rangle\langle x'-\delta x|\psi\rangle dx'$$

$$x' = x+\delta x$$

$$\langle x'-\delta x|\psi\rangle = \psi(x'-\delta x) \cong \psi(x') - \delta x\cdot\frac{\partial}{\partial x'}\psi(x') = \langle x'|\psi\rangle - \delta x\cdot\frac{\partial}{\partial x'}\langle x'|\psi\rangle$$

$$\hat{T}(\delta x)|\psi\rangle = \int |x'\rangle\left(\langle x'|\psi\rangle - \delta x\frac{\partial}{\partial x'}\langle x'|\psi\rangle\right)dx'$$

$$= \int |x'\rangle\langle x'|\psi\rangle dx' - \delta x\int |x'\rangle\frac{\partial}{\partial x'}\langle x'|\psi\rangle dx' = |\psi\rangle - \delta x\int |x'\rangle\frac{\partial}{\partial x'}\langle x'|\psi\rangle dx'$$

$$= \left(1-\frac{i}{\hbar}\hat{p}_x\,\delta x\right)|\psi\rangle = |\psi\rangle - \frac{i}{\hbar}\hat{p}_x\,\delta x|\psi\rangle$$

$$-\frac{i}{\hbar}\hat{p}_x\,\delta x\,|\psi\rangle = -\delta x\int |x'\rangle \frac{\partial}{\partial x'}\langle x'|\psi\rangle\,dx'$$

$$\langle x|\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\int \langle x|x'\rangle \frac{\partial}{\partial x'}\langle x'|\psi\rangle\,dx'$$

$$\langle x|\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$$

$$\hat{p}_x \xrightarrow[\text{x Basis}]{} \frac{\hbar}{i}\frac{\partial}{\partial x}$$

Momentum Space

$$\hat{p}_x |p\rangle = p|p\rangle$$

$$\langle p'|p\rangle = \delta(p-p')$$

$$P(p_1,p_2) = \int_{p_1}^{p_2} |\langle p|\psi\rangle|^2 dp$$

$$\langle p|\psi\rangle = \psi(p)$$

$$|\psi\rangle = \underbrace{\int |p\rangle\langle p|}_{\hat{1}}\psi\rangle\, dp$$

$$\int_{-\infty}^{\infty} |\langle p|\psi\rangle|^2 dp = 1$$

$$\langle x|\hat{p}_x|p\rangle = \langle x|p|p\rangle \qquad = p\langle x|p\rangle$$

$$\langle x|\hat{p}_x|p\rangle = p\langle x|p\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|p\rangle$$

$$\frac{\partial}{\partial x}\psi(x) = \frac{ip}{\hbar}\psi(x) \qquad \psi(x) = N e^{\frac{ipx}{\hbar}}$$

$$\langle p'|p\rangle = \int \langle p'|x\rangle\langle x|p\rangle dx = NN^{*}\int dx\, e^{\frac{-ip'x}{\hbar}} \cdot e^{\frac{ipx}{\hbar}}$$

$$\langle p'|p\rangle = |N|^2 \int e^{\frac{i(p-p')x}{\hbar}} dx = |N|^2 \cdot 2\pi\hbar\, \delta(p-p')$$

$$N = \frac{1}{\sqrt{2\pi\hbar}}$$

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}} \qquad \langle p | x \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-ipx}{\hbar}}$$

$$e^{\frac{ipx}{\hbar}} = \cos\left(\frac{px}{\hbar}\right) + i\sin\left(\frac{px}{\hbar}\right)$$

$$\lambda = \frac{2\pi\hbar}{p} = \frac{h}{p}$$ ← de Broglie wavelength