Quantum Mechanics I (PHYS 3143A) Fall 2020

Test 2 **Name:**

Problem 1

A spin-2 particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|2,-2\rangle_z + \frac{1}{\sqrt{3}}|2,-1\rangle_z + \frac{1}{\sqrt{3}}|2,2\rangle_z$$

a) (10 points) Evaluate the uncertainty ΔS_z

$$\langle S_z\rangle = (-2\hbar)\cdot\frac{1}{3} + (-\hbar)\cdot\frac{1}{3} + (2\hbar)\cdot\frac{1}{3} = -\frac{2\hbar}{3} - \frac{\hbar}{3} + \frac{2\hbar}{3} = -\frac{\hbar}{3}$$

$$\langle S_z^2\rangle = (-2\hbar)^2\cdot\frac{1}{3} + (-\hbar)^2\cdot\frac{1}{3} + (2\hbar)^2\cdot\frac{1}{3} = \frac{4\hbar^2}{3} + \frac{\hbar^2}{3} + \frac{4\hbar^2}{3} = \frac{9\hbar^2}{3} = 3\hbar^2$$

$$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{3\hbar^2 - \left(-\frac{\hbar}{3}\right)^2} = \sqrt{\frac{27\hbar^2}{9} - \frac{\hbar^2}{9}} = \boxed{\frac{\hbar}{3}\sqrt{26}}$$

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y$$
$$\hat{S}_- = \hat{S}_x - i\hat{S}_y$$
$$\hat{S}_+ - \hat{S}_- = 2i\hat{S}_y$$

$$\hat{S}_y = \frac{1}{2i}\left(\hat{S}_+ - \hat{S}_-\right)$$

$$\hat{S}_+|s,m\rangle = \sqrt{s(s+1)-m(m+1)}\,\hbar|s,m+1\rangle$$
$$\hat{S}_-|s,m\rangle = \sqrt{s(s+1)-m(m-1)}\,\hbar|s,m-1\rangle$$

b) (25 points) Calculate $\langle \hat{S}_y \rangle$ and $\langle \hat{S}_y^2 \rangle$ for this state and evaluate the uncertainty ΔS_y. Note: determining the matrix representation of $\hat{S}_y$ operator is not necessary in this problem.

$$\hat{S}_y|\psi\rangle = \frac{1}{2i}\left(\hat{S}_+ - \hat{S}_-\right)\left(\frac{1}{\sqrt{3}}|\underset{s}{2},\underset{m}{-2}\rangle + \frac{1}{\sqrt{3}}|\underset{s}{2},\underset{m}{-1}\rangle + \frac{1}{\sqrt{3}}|\underset{s}{2},\underset{m}{2}\rangle\right) =$$

$$= \frac{1}{2i}\Bigg(\frac{1}{\sqrt{3}}\underbrace{\sqrt{\underbrace{2(2+1)}_{6}-\underbrace{(-2)(-2+1)}_{-2}}}_{2}\,\hbar|2,-1\rangle + \frac{1}{\sqrt{3}}\underbrace{\sqrt{2(2+1)-(-1)(-1+1)}}_{\sqrt{6}}\,\hbar|2,0\rangle + \frac{1}{\sqrt{3}}\underbrace{\cancel{\sqrt{2(2+1)-2(2+1)}\,\hbar|2,?\rangle}}_{0} -$$

$$-\frac{1}{\sqrt{3}}\underbrace{\sqrt{2(2+1)-(-1)(-1-1)}}_{2}\,\hbar|2,-2\rangle - \frac{1}{\sqrt{3}}\underbrace{\sqrt{2(2+1)-2(2-1)}}_{2}\,\hbar|2,1\rangle\Bigg) =$$

$$= \frac{1}{2i}\left(\frac{2}{\sqrt{3}}\hbar|2,-1\rangle + \frac{\sqrt{6}}{\sqrt{3}}\hbar|2,0\rangle - \frac{2}{\sqrt{3}}\hbar|2,-2\rangle - \frac{2}{\sqrt{3}}\hbar|2,1\rangle\right) =$$

$$\hat{S}_y|\psi\rangle = \boxed{\frac{\hbar}{2i\sqrt{3}}\left(2|2,-1\rangle + \sqrt{6}|2,0\rangle - 2|2,-2\rangle - 2|2,1\rangle\right)}$$

$$\langle S_y\rangle = \langle\psi|\underbrace{\hat{S}_y|\psi\rangle} = \underbrace{\frac{1}{\sqrt{3}}\left(\langle 2,-2| + \langle 2,-1| + \langle 2,2|\right)}_{\langle\psi|}\underbrace{\frac{\hbar}{2i\sqrt{3}}\left(2|2,-1\rangle + \sqrt{6}|2,0\rangle - 2|2,-2\rangle - 2|2,1\rangle\right)}_{\hat{S}_y|\psi\rangle}$$

$$= \frac{\hbar}{2i\sqrt{3}\cdot\sqrt{3}}\left(-2 + 2\right) = 0$$

$$\langle S_y^2\rangle = \left(\langle\psi|\hat{S}_y\right)\left(\hat{S}_y|\psi\rangle\right) =$$

$$= \underbrace{\frac{\hbar}{-2i\sqrt{3}}\left(2\langle 2,-1| + \sqrt{6}\langle 2,0| - 2\langle 2,-2| - 2\langle 2,1|\right)}_{\langle\psi|\hat{S}_y}\underbrace{\frac{\hbar}{2i\sqrt{3}}\left(2|2,-1\rangle + \sqrt{6}|2,0\rangle - 2|2,-2\rangle - 2|2,1\rangle\right)}_{\hat{S}_y|\psi\rangle}$$

$$= \frac{\hbar^2}{4\cdot 3}\left(4 + 6 + 4 + 4\right) = \frac{18\hbar^2}{4\cdot 3} = \frac{3\hbar^2}{2}$$

$$\Delta S_y = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} = \sqrt{\frac{3\hbar^2}{2} - 0} = \boxed{\hbar\sqrt{\frac{3}{2}}}$$

Test 2_sample

$\hat{S}_z|s,m\rangle = \hbar m|s,m\rangle$

$\hat{S}^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$

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Test 2 **Name:**

$|s,m\rangle$

Problem 1

A spin-3 particle is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}|3,1\rangle + \frac{1}{\sqrt{2}}|3,2\rangle$.

a) (10 points) Calculate $\langle S_z\rangle$ and $\langle S_z^2\rangle$ for this state and evaluate the uncertainty ΔS_z

$$\langle S_z\rangle = \hbar\cdot\frac{1}{2} + 2\hbar\cdot\frac{1}{2} = \frac{3\hbar}{2}$$

$$\langle S_z^2\rangle = \hbar^2\cdot\frac{1}{2} + (2\hbar)^2\cdot\frac{1}{2} = \frac{\hbar^2}{2} + \frac{4\hbar^2}{2} = \frac{5\hbar^2}{2}$$

$$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \sqrt{\frac{5\hbar^2}{2} - \left(\frac{3\hbar}{2}\right)^2} = \sqrt{\frac{5\hbar^2\cdot 2}{2\cdot 2} - \frac{9\hbar^2}{4}} = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2}$$

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \qquad \hat{S}_+|s,m\rangle = \sqrt{s(s+1)-m(m+1)}\,\hbar\,|s,m+1\rangle$$

$$\hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) \qquad \hat{S}_-|s,m\rangle = \sqrt{s(s+1)-m(m-1)}\,\hbar\,|s,m-1\rangle$$

b) (35 points) Calculate $\langle S_x\rangle$ and $\langle S_x^2\rangle$ for this state and evaluate the uncertainty ΔS_x. Note: determining the matrix representation of $\hat{S}_x$ operator is not necessary in this problem.

$$\hat{S}_x|\psi\rangle = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)\left(\frac{1}{\sqrt{2}}|3,1\rangle + \frac{1}{\sqrt{2}}|3,2\rangle\right) = \frac{1}{2}\Bigg[\frac{1}{\sqrt{2}}\sqrt{\underbrace{3(3+1)}_{12}-\underbrace{1(1+1)}_{2}}\,\hbar\,|3,2\rangle \; +$$

$$+\frac{1}{\sqrt{2}}\sqrt{\underbrace{3(3+1)}_{12}-\underbrace{2(2+1)}_{6}}\,\hbar\,|3,3\rangle + \frac{1}{\sqrt{2}}\sqrt{\underbrace{3(3+1)}_{12}-\underbrace{1(1-1)}_{0}}\,\hbar\,|3,0\rangle + \frac{1}{\sqrt{2}}\sqrt{\underbrace{3(3+1)}_{12}-\underbrace{2(2-1)}_{2}}\,\hbar\,|3,1\rangle\Bigg] =$$

$$\hat{S}_x|\psi\rangle = \underbrace{\frac{1}{2}\left[\sqrt{5}\,\hbar\,|3,2\rangle + \sqrt{3}\,\hbar\,|3,3\rangle + \sqrt{6}\,\hbar\,|3,0\rangle + \sqrt{5}\,\hbar\,|3,1\rangle\right]}$$

$$\langle S_x\rangle = \langle\psi|\underbrace{\hat{S}_x|\psi\rangle} = \overbrace{\left(\frac{1}{\sqrt{2}}\langle 3,1| + \frac{1}{\sqrt{2}}\langle 3,2|\right)}^{\langle\psi|}\frac{\hbar}{2}\left(\sqrt{5}|3,2\rangle + \sqrt{3}|3,3\rangle + \sqrt{6}|3,0\rangle + \sqrt{5}|3,1\rangle\right) =$$

$$= \frac{\hbar}{2}\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{5} + \frac{\hbar}{2}\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{5} = \hbar\sqrt{\frac{5}{2}}$$

$$\langle S_x^2\rangle = \left(\langle\psi|\hat{S}_x\right)\cdot\left(\hat{S}_x|\psi\rangle\right) =$$

$$\frac{\hbar}{2}\left(\sqrt{5}\langle 3,2| + \sqrt{3}\langle 3,3| + \sqrt{6}\langle 3,0| + \sqrt{5}\langle 3,1|\right)\cdot\frac{\hbar}{2}\left(\sqrt{5}|3,2\rangle + \sqrt{3}|3,3\rangle + \sqrt{6}|3,0\rangle + \sqrt{5}|3,1\rangle\right) =$$

$$= \frac{\hbar^2}{4}(5 + 3 + 6 + 5) = \frac{19\hbar^2}{4}$$

$$\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \sqrt{\frac{19\hbar^2}{4} - \left(\hbar\sqrt{\frac{5}{2}}\right)^2} = \sqrt{\frac{19\hbar^2}{4} - \frac{5\hbar^2\cdot 2}{2\cdot 2}} =$$

$$= \sqrt{\frac{9\hbar^2}{4}} = \frac{3\hbar}{2}$$

Problem 2.

The Hamiltonian for a spin-1 particle is represented by the matrix

$$\hat{H} \xrightarrow{S_z\, basis} \hbar\omega_0 \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

A particle is initially in the state

$$|\psi(0)\rangle \xrightarrow{S_z\, basis} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

a) (40 points) Determine the state of the system $|\psi(t)\rangle$ at time t in S_z basis.

1) Find eigenvalues of $\hat{H}$:

$$\det \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & 2 \\ 2 & 1-\lambda \\ 0 & 0 \end{matrix} = (1-\lambda)(1-\lambda)(2-\lambda) - 2\cdot 2(2-\lambda) = 0$$

$$(1-\lambda)^2(2-\lambda) - 4(2-\lambda) = 0$$

$$(2-\lambda)\left((1-\lambda)^2 - 2^2\right) = 0$$

$$(2-\lambda)(1-\lambda-2)(1-\lambda+2) = 0$$

$$(2-\lambda)(-1-\lambda)(3-\lambda) = 0$$

$$\lambda = 2, -1, 3$$

Eigenvalues of $\hat{H}$:

$2\hbar\omega_0, -\hbar\omega_0, 3\hbar\omega_0$

2) Find eigenstates of $\hat{H}$:

$|2\hbar\omega_0\rangle$: $\lambda = 2$

$$\begin{pmatrix} 1-2 & 2 & 0 \\ 2 & 1-2 & 0 \\ 0 & 0 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_2 = 0$

$2x_1 - x_2 = 0 \Rightarrow 2(2x_2) - x_2 = 0 \Rightarrow 3x_2 = 0$

$x_2 = 0$

$$|2\hbar\omega_0\rangle \xrightarrow{S_z\, Basis} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$c^*(0,0,1)\cdot c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$

$c^* c\cdot(0+0+1) = 1$

$|c|^2 = 1$

$c = 1$

$|-\hbar\omega_0\rangle$ $\lambda = -1$

$$\begin{pmatrix} 1-(-1) & 2 & 0 \\ 2 & 1-(-1) & 0 \\ 0 & 0 & 2-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 2-(-1) \end{pmatrix}\begin{pmatrix} \cdot \\ x_3 \end{pmatrix} = \begin{pmatrix} \cdot \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} \cdot \\ x_3 \end{pmatrix} = \begin{pmatrix} \cdot \\ 0 \end{pmatrix}$$

$2x_1 + 2x_2 = 0 \Rightarrow x_1 = -x_2$

$3x_3 = 0 \Rightarrow x_3 = 0$

$$|-\hbar\omega_0\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \overset{c}{\frac{1}{\sqrt{2}}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$c^*(1,-1,0)c\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1$

$c^* c\,(1+1+0) = 1$

$|c|^2 = \frac{1}{2}$

$c = \frac{1}{\sqrt{2}}$

$|3\hbar\omega\rangle \qquad \lambda = 3$

$$\begin{pmatrix} 1-3 & 2 & 0 \\ 2 & 1-3 & 0 \\ 0 & 0 & 2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x_1 + 2x_2 = 0 \Rightarrow x_1 = x_2$

$-x_3 = 0 \Rightarrow x_3 = 0$

$$|3\hbar\omega_0\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

3) Express $|\psi(0)\rangle$ in the basis of eigenstates of $\hat{H}$:

$$|\psi(0)\rangle = |2\hbar\omega_0\rangle\underbrace{\langle 2\hbar\omega_0|\psi(0)\rangle}_{0} + |-\hbar\omega_0\rangle\underbrace{\langle -\hbar\omega_0|\psi(0)\rangle}_{\frac{1}{\sqrt{2}}} + |3\hbar\omega_0\rangle\underbrace{\langle 3\hbar\omega_0|\psi(0)\rangle}_{\frac{1}{\sqrt{2}}}$$

$$\langle 2\hbar\omega_0|\psi(0)\rangle = (0,0,1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\langle -\hbar\omega_0|\psi(0)\rangle = \frac{1}{\sqrt{2}}(1,-1,0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle 3\hbar\omega_0|\psi(0)\rangle = \frac{1}{\sqrt{2}}(1,1,0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\boxed{|\psi(0)\rangle = \frac{1}{\sqrt{2}}|-\hbar\omega_0\rangle + \frac{1}{\sqrt{2}}|3\hbar\omega_0\rangle}$$

4) Evolve $|\psi(0)\rangle$

$$|E(t)\rangle = e^{-\frac{iEt}{\hbar}}|E(0)\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{\frac{-i(-\hbar\omega_0)t}{\hbar}}|-\hbar\omega_0\rangle + \frac{1}{\sqrt{2}} e^{\frac{-i(3\hbar\omega_0)t}{\hbar}}|3\hbar\omega_0\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{i\omega_0 t}|-\hbar\omega_0\rangle + \frac{1}{\sqrt{2}} e^{-3i\omega_0 t}|3\hbar\omega_0\rangle$$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

5) Switch to original basis (S_z basis)

$$|\psi(t)\rangle \xrightarrow[S_z \text{ basis}]{} \frac{1}{\sqrt{2}} e^{i\omega_0 t} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} e^{-3i\omega_0 t} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{i\omega_0 t} + e^{-3i\omega_0 t} \\ -e^{i\omega_0 t} + e^{-3i\omega_0 t} \\ 0 \end{pmatrix}$$

$$= e^{-i\omega_0 t}\begin{pmatrix} \frac{e^{2i\omega_0 t} + e^{-2i\omega_0 t}}{2} \\ \frac{-e^{2i\omega_0 t} + e^{-2i\omega_0 t}}{2(-i)}(-i) \end{pmatrix} = \cancel{e^{-i\omega_0 t}}\begin{pmatrix} \cos(2\omega_0 t) \\ -i \cdot \sin(2\omega_0 t) \end{pmatrix}$$

$$= e \begin{pmatrix} \frac{-e^{2i\omega_0 t} + e^{-2i\omega_0 t}}{2(-i)} (i) \\ 0 \end{pmatrix} = \cancel{e} \begin{pmatrix} -i \cdot \sin(2\omega_0 t) \\ 0 \end{pmatrix}$$

$$|\psi(t)\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \cos(2\omega_0 t) \\ -i \cdot \sin(2\omega_0 t) \\ 0 \end{pmatrix}$$

b) (15 points) Calculate the expectation value $\langle S_x \rangle$ as a function of time.

$$\hat{S}_x \xrightarrow[S_z\ \text{basis}]{} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\langle S_x \rangle = \langle \psi(t) | \hat{S}_x | \psi(t) \rangle = (\cos(2\omega_0 t), i\sin(2\omega_0 t), 0) \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(2\omega_0 t) \\ -i\sin(2\omega_0 t) \\ 0 \end{pmatrix}$$

$$= (\cos(2\omega_0 t), i\sin(2\omega_0 t), 0) \cdot \frac{\hbar}{\sqrt{2}} \begin{pmatrix} -i\cdot\sin(2\omega_0 t) \\ \cos(2\omega_0 t) \\ -i\sin(2\omega_0 t) \end{pmatrix} =$$

$$= \frac{\hbar}{\sqrt{2}} \left(\underbrace{\cancel{\cos(2\omega_0 t) \cdot (-i) \sin(2\omega_0 t)}} + \underbrace{\cancel{i\sin(2\omega_0 t) \cdot \cos(2\omega_0 t)}} + 0 \right)$$

$$= 0.$$

$$\langle S_y \rangle = \langle \psi(t) | \hat{S}_y | \psi(t) \rangle = (\cos(2\omega_0 t), i\cdot\sin(2\omega_0 t), 0) \cdot \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \cos(2\omega_0 t) \\ -i\sin(2\omega_0 t) \\ 0 \end{pmatrix} =$$

$$= (\cos(2\omega_0 t), i\sin(2\omega_0 t), 0) \cdot \frac{\hbar}{\sqrt{2}} \begin{pmatrix} -\sin(2\omega_0 t) \\ i\cos(2\omega_0 t) \\ \sin(2\omega_0 t) \end{pmatrix} =$$

$$= \frac{\hbar}{\sqrt{2}} \left(-\cos(2\omega_0 t) \cdot \sin(2\omega_0 t) - \sin(2\omega_0 t)\cos(2\omega_0 t) \right) =$$

$$= -\frac{\hbar}{\sqrt{2}} \cdot \underbrace{2\cos(2\omega_0 t) \cdot \sin(2\omega_0 t)}_{\sin(4\omega_0 t)} = -\frac{\hbar}{\sqrt{2}} \sin(4\omega_0 t)$$