Time evolution of $|\psi(0)\rangle$

1) Find Eigenvalues and Eigenvectors of $\hat{H}$:

$E_1, E_2, \ldots$ $\quad |E_1\rangle, |E_2\rangle, \ldots$

2) Express $|\psi(0)\rangle$ in the basis of $|E_1\rangle, |E_2\rangle, \ldots$

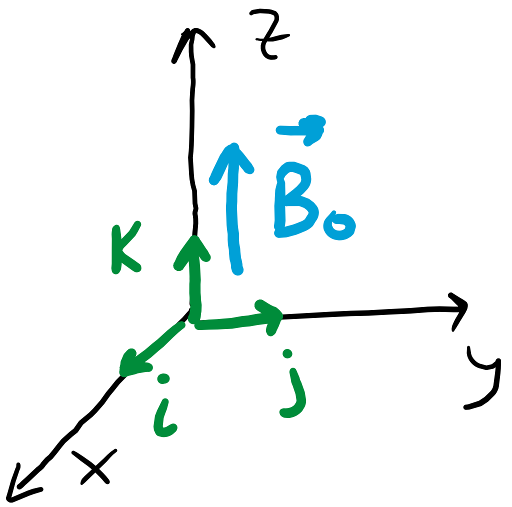
$$|\psi(0)\rangle = |E_1\rangle\langle E_1|\psi(0)\rangle + |E_2\rangle\langle E_2|\psi(0)\rangle + \ldots$$

3) Time evolution:

$$|\psi(t)\rangle = e^{-\frac{iE_1 t}{\hbar}}|E_1\rangle\langle E_1|\psi(0)\rangle + e^{-\frac{iE_2 t}{\hbar}}|E_2\rangle\langle E_2|\psi(0)\rangle + \ldots$$

4) Express $|\psi(t)\rangle$ in original basis

Spin ½ particle in Magnetic Field



$$\hat{H} = -\hat{\mu}\cdot\vec{B} = -\frac{gq}{2mc}\hat{S}\cdot\vec{B} = \frac{ge}{2mc}\hat{S}_z \cdot B_0$$

$$\vec{B} = B_0\vec{K}$$

$$\hat{H} = \overbrace{\frac{geB_0}{2mc}}^{\omega_0}\hat{S}_z = \omega_0\hat{S}_z$$

$$\hat{H}|+z\rangle = \omega_0\hat{S}_z|+z\rangle = \overbrace{\frac{\hbar\omega_0}{2}}^{E_+}|+z\rangle = E_+|+z\rangle$$

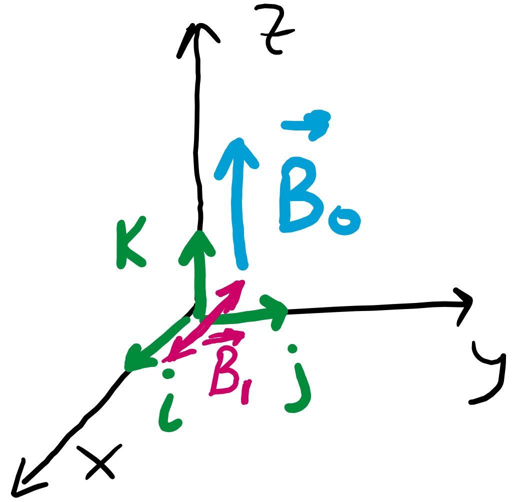
$$\hat{H}|-z\rangle = \omega_0\hat{S}_z|-z\rangle = \overbrace{-\frac{\hbar\omega_0}{2}}^{E_-}|-z\rangle = E_-|-z\rangle$$

$$\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}} = e^{-\frac{i\omega_0 t\hat{S}_z}{\hbar}} = \hat{R}(\omega_0 t\vec{K})$$

$$\hat{R}(\phi\vec{K}) = e^{-\frac{i\phi\hat{S}_z}{\hbar}}$$

$$|\psi(0)\rangle = |+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

$$|\psi(t)\rangle = e^{-\frac{iE_+t}{\hbar}} \cdot \frac{1}{\sqrt{2}}|+z\rangle + e^{-\frac{iE_-t}{\hbar}} \cdot \frac{1}{\sqrt{2}}|-z\rangle = e^{-i\frac{\omega_0 t}{2}} \cdot \frac{1}{\sqrt{2}}|+z\rangle + e^{\frac{i\omega_0 t}{2}} \cdot \frac{1}{\sqrt{2}}|-z\rangle$$



$$\hat H = -\hat\mu\cdot\vec B = -\frac{qg}{2mc}\hat S\cdot\vec B$$

$$= \frac{eg}{2mc}\left(\hat S_x\vec i + \hat S_y\vec j + \hat S_z\vec k\right)\left(B_0\vec K + B_1\cos(\omega t)\vec i\right)$$

$$= \underbrace{\frac{eg B_1}{2mc}}_{\omega_1}\cos(\omega t)\hat S_x + \underbrace{\frac{eg}{2mc}B_0}_{\omega_0}\hat S_z$$

$$\vec B = B_0\vec K + B_1\cos(\omega t)\vec i$$

$$B_1 \ll B_0 \qquad \omega_1 \ll \omega_0$$

$$\hat H = \omega_0\hat S_z + \omega_1\cos(\omega t)\hat S_x$$

$$\hat H \xrightarrow[S_z\text{ basis}]{} \omega_0\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \omega_1\cos(\omega t)\cdot\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\omega_0 & \omega_1\cos\omega t\\ \omega_1\cos(\omega t) & -\omega_0\end{pmatrix}$$

$$|\psi(t)\rangle \xrightarrow[S_z\ \text{Basis}]{} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \qquad |\psi(0)\rangle \xrightarrow[S_z\ \text{Basis}]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1\cos\omega t \\ \omega_1\cos(\omega t) & -\omega_0 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = i\hbar\begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix} \iff \begin{matrix} \frac{1}{2}\omega_0 a + \frac{1}{2}\omega_1\cos(\omega t)b = i\dot{a} \\ \frac{1}{2}\omega_1\cos(\omega t)a - \frac{1}{2}\omega_0 b = i\dot{b} \end{matrix}$$

If $\omega_1 = 0 \Rightarrow \begin{matrix} \dot{a} = -\frac{i\omega_0}{2}a \\ \dot{b} = +\frac{i\omega_0}{2}b \end{matrix} \Rightarrow \begin{matrix} a(t) = a(0)e^{-\frac{i\omega_0}{2}t} \\ b(t) = b(0)e^{-\frac{i\omega_0 t}{2}} \end{matrix}$

$$a(t) = c(t)e^{-\frac{i\omega_0 t}{2}}$$

$$b(t) = d(t)e^{\frac{i\omega_0 t}{2}}$$

$$\dot{a} = \dot{c}e^{-\frac{i\omega_0 t}{2}} - \frac{i\omega_0}{2}ce^{-\frac{i\omega_0 t}{2}}$$

$$\dot{b} = \dot{d}e^{\frac{i\omega_0 t}{2}} + \frac{i\omega_0}{2}de^{\frac{i\omega_0 t}{2}}$$

$$\Rightarrow \frac{1}{2}\omega_0 ce^{-\frac{i\omega_0 t}{2}} + \frac{1}{2}\omega_1\cos(\omega t)\cdot de^{\frac{i\omega_0 t}{2}} = i\dot{c}e^{-\frac{i\omega_0 t}{2}} + \frac{\omega_0}{2}ce^{-\frac{i\omega_0 t}{2}}$$

$$\frac{1}{2}\omega_1\cos(\omega t)\cdot ce^{-\frac{i\omega_0 t}{2}} - \frac{1}{2}\omega_0 de^{\frac{i\omega_0 t}{2}} = i\dot{d}e^{\frac{i\omega_0 t}{2}} - \frac{\omega_0}{2}de^{\frac{i\omega_0 t}{2}}$$

$$i\dot{c} = \frac{\omega_1}{2}\cos(\omega t)\cdot d\cdot e^{i\omega_0 t}$$

$$i\dot{d} = \frac{\omega_1}{2}\cos(\omega t)\cdot c\cdot e^{-i\omega_0 t}$$

$$\Rightarrow \quad i\dot{c} = \frac{\omega_1}{4}\left(e^{i(\omega_0+\omega)t} + e^{i(\omega_0-\omega)t}\right)\cdot d$$

$$i\dot{d} = \frac{\omega_1}{4}\left(e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}\right)\cdot c$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

1) $$i\dot{c} = \frac{\omega_1}{4} e^{i(\omega_0-\omega)t}\cdot d$$

$$i\dot{d} = \frac{\omega_1}{4} e^{i(\omega-\omega_0)t}\cdot c$$

2) $\omega_0 = \omega$

$$i\dot{c} = \frac{\omega_1}{4}\cdot d$$

$$i\dot{d} = \frac{\omega_1}{4}\cdot c$$

$$\Rightarrow \quad i\ddot{c} = \frac{\omega_1}{4}\dot{d} = \frac{\omega_1}{4}\left(-i\frac{\omega_1}{4}c\right)$$

$$i\ddot{d} = \frac{\omega_1}{4}\dot{c} = \frac{\omega_1}{4}\left(-i\frac{\omega_1}{4}d\right)$$

$$\ddot{c} = -\left(\frac{\omega_1}{4}\right)^2 c$$

$$\ddot{d} = -\left(\frac{\omega_1}{4}\right)^2 d$$

For $|\psi(0)\rangle \xrightarrow[S_z \text{ Basis}]{} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$c(0) = 1$$

$$d(0) = 0$$

$$c(t) = \cos\left(\frac{\omega_1}{4}t\right)$$

$$d(t) = -i\sin\left(\frac{\omega_1}{4}t\right)$$

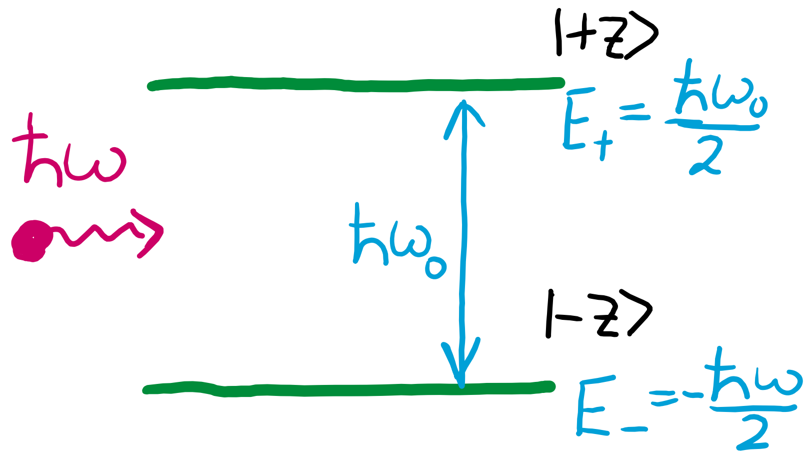
$$|\psi(t)\rangle \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \cos\left(\frac{\omega_1}{4}t\right) e^{-\frac{i\omega_0 t}{2}} \\ -i\sin\left(\frac{\omega_1 t}{4}\right) e^{\frac{i\omega_0 t}{2}} \end{pmatrix}$$

$$|\langle +z|\psi(t)\rangle|^2 = \cos^2\left(\frac{\omega_1 t}{4}\right)$$

$$|\langle -z|\psi(t)\rangle|^2 = \sin^2\left(\frac{\omega_1 t}{4}\right)$$

More accurate solution (Rabi formula):

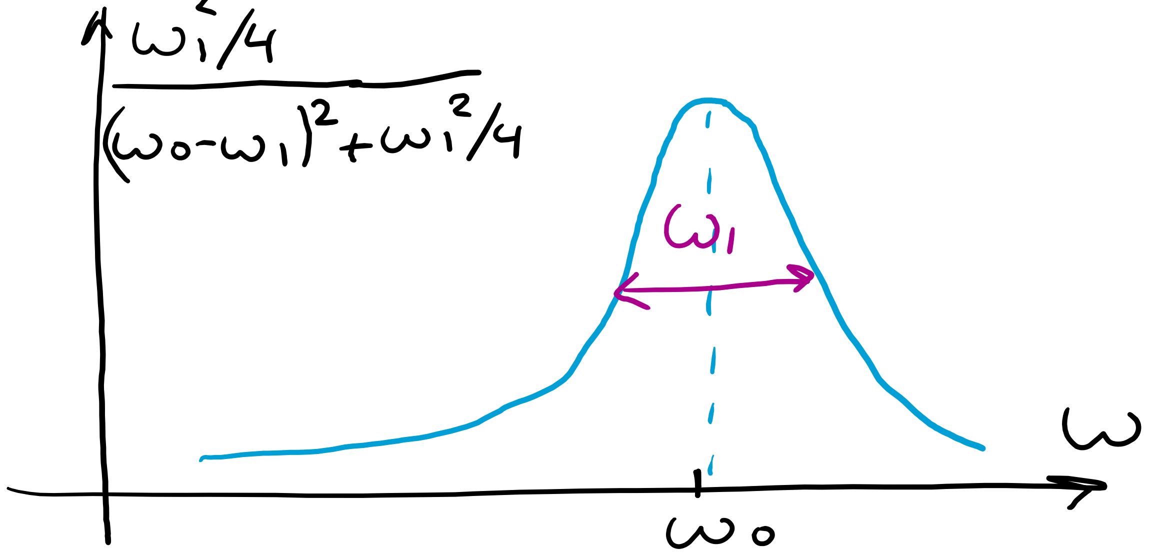
$$|\langle -z|\psi(t)\rangle|^2 = \frac{\omega_1^2/4}{(\omega_0-\omega)^2+\omega_1^2/4}\sin^2\left(\frac{\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}}{2}t\right)$$



$$\Delta E \approx \frac{\omega_1}{2}\hbar$$

$$\Delta t \approx \frac{1}{\omega_1}$$

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{2}$$



Lasers

MRI

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