Time Evolution

$$\hat{U}(t)|\psi(0)\rangle = |\psi(t)\rangle$$

$$\langle\psi(0)|\hat{U}^{+}(t) = \langle\psi(t)|$$

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\underbrace{\hat{U}^{+}(t)\,\hat{U}(t)}_{\hat{1}}|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

$$\hat{U}^{+}(t)\hat{U}(t) = \hat{1}$$

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}dt$$

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}dt$$

$$\hat{U}(t+dt) = \hat{U}(dt)\hat{U}(t) = \left(1 - \frac{i}{\hbar}\hat{H}dt\right)\hat{U}(t)$$

$$\hat{U}(t+dt) - \hat{U}(t) = -\frac{idt}{\hbar}\hat{H}\hat{U}(t)$$

$$\frac{\hat{U}(t+dt) - \hat{U}(t)}{dt} = \frac{d}{dt}\hat{U}(t) = -\frac{i}{\hbar}\hat{H}\hat{U}(t)$$

$$i\hbar\frac{d}{dt}\hat{U}(t)|\psi(0)\rangle = \hat{H}\hat{U}(t)|\psi(0)\rangle$$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}\,dt$$

$$\hat{U}(t) = \lim_{N\to\infty}\left(1 - \frac{i}{\hbar}\hat{H}\frac{t}{N}\right)^N = e^{-\frac{i\hat{H}t}{\hbar}}$$

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$$

$$\langle\psi(t)|\hat{H}|\psi(t)\rangle = \langle\psi(0)|\hat{U}^+(t)\,\hat{H}\,\hat{U}(t)|\psi(0)\rangle = \langle\psi(0)|\hat{H}\hat{U}^+(t)\hat{U}(t)|\psi(0)\rangle = \langle\psi(0)|\hat{H}|\psi(0)\rangle$$

Erwin Schrödinger

$$\langle \psi(t)|\hat{H}|\psi(t)\rangle = \langle \psi(0)|\hat{H}|\psi(0)\rangle = \langle E\rangle$$

$$\hat{H}|E\rangle = E|E\rangle \longleftarrow \text{stationary state}$$

$$\hat{U}(t)|E\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|E\rangle = \left[1 - \frac{i\hat{H}t}{\hbar} + \frac{1}{2!}\left(-\frac{i\hat{H}t}{\hbar}\right)^2 + \frac{1}{3!}\left(-\frac{i\hat{H}t}{\hbar}\right)^3 + \dots\right]|E\rangle$$

$$= \left[1 - \frac{iEt}{\hbar} + \frac{1}{2!}\left(-\frac{iEt}{\hbar}\right)^2 + \frac{1}{3!}\left(-\frac{iEt}{\hbar}\right)^3 + \dots\right]|E\rangle = e^{-\frac{iEt}{\hbar}}|E\rangle$$

$$|\psi(0)\rangle = |E\rangle \Rightarrow |\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle = e^{-\frac{iEt}{\hbar}}|E\rangle$$

$$\langle A \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle$$

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} \hat{H} |\psi(t)\rangle$$

$$\frac{d}{dt} \langle \psi(t)| = \frac{i}{\hbar} \langle \psi(t)| \hat{H}$$

$$\frac{d}{dt}\langle A \rangle = \underbrace{\left(\frac{d}{dt}\langle \psi(t)|\right)}_{\frac{i}{\hbar}\langle \psi(t)|\hat{H}} \hat{A}|\psi(t)\rangle + \langle \psi(t)|\left(\frac{\partial}{\partial t}\hat{A}\right)|\psi(t)\rangle + \langle \psi(t)|\hat{A}\underbrace{\left(\frac{d}{dt}|\psi(t)\rangle\right)}_{-\frac{i}{\hbar}\hat{H}|\psi(t)\rangle}$$

$$= \frac{i}{\hbar}\langle \psi(t)|\hat{H}\hat{A}|\psi(t)\rangle - \frac{i}{\hbar}\langle \psi(t)|\hat{A}\hat{H}|\psi(t)\rangle + \langle \psi(t)|\left(\frac{\partial}{\partial t}\hat{A}\right)|\psi(t)\rangle$$

$$= \frac{i}{\hbar}\langle \psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle + \langle \psi(t)|\left(\frac{\partial}{\partial t}\hat{A}\right)|\psi(t)\rangle$$

Time evolution of $|\psi(0)\rangle$

1) Find Eigenvalues and Eigenvectors of $\hat{H}$:

$E_1, E_2, \ldots$ $\qquad$ $|E_1\rangle, |E_2\rangle, \ldots$

2) Express $|\psi(0)\rangle$ in the basis of $|E_1\rangle, |E_2\rangle, \ldots$

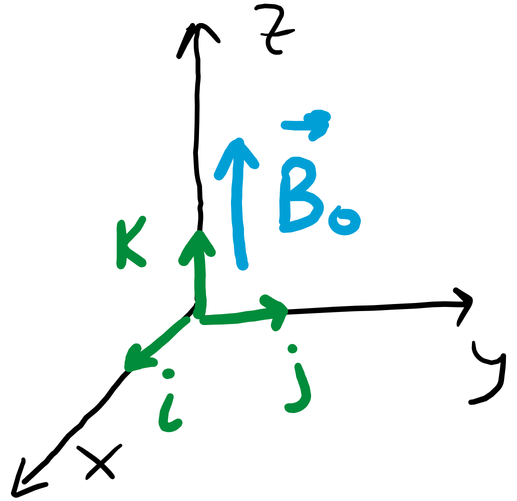
$$|\psi(0)\rangle = |E_1\rangle\langle E_1|\psi(0)\rangle + |E_2\rangle\langle E_2|\psi(0)\rangle + \ldots$$

3) Time evolution:

$$|\psi(t)\rangle = e^{-\frac{iE_1 t}{\hbar}}|E_1\rangle\langle E_1|\psi(0)\rangle + e^{-\frac{iE_2 t}{\hbar}}|E_2\rangle\langle E_2|\psi(0)\rangle + \ldots$$

4) Express $|\psi(t)\rangle$ in original basis

Spin ½ particle in Magnetic Field



$$\hat{H} = -\hat{\mu}\cdot\vec{B} = -\frac{gq}{2mc}\hat{S}\cdot\vec{B} = \frac{ge}{2mc}\hat{S}_z \cdot B_0$$

$$\vec{B} = B_0\vec{K}$$

$$\hat{H} = \overbrace{\frac{geB_0}{2mc}}^{\omega_0}\hat{S}_z = \omega_0\hat{S}_z$$

$$\hat{H}|+z\rangle = \omega_0\hat{S}_z|+z\rangle = \overbrace{\frac{\hbar\omega_0}{2}}^{E_+}|+z\rangle = E_+|+z\rangle$$

$$\hat{H}|-z\rangle = \omega_0\hat{S}_z|-z\rangle = \overbrace{-\frac{\hbar\omega_0}{2}}^{E_-}|-z\rangle = E_-|-z\rangle$$

$$\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}} = e^{-\frac{i\omega_0 t\hat{S}_z}{\hbar}} = \hat{R}(\omega_0 t\vec{K})$$

$$\hat{R}(\phi\vec{K}) = e^{-\frac{i\phi\hat{S}_z}{\hbar}}$$

$$|\psi(0)\rangle = |+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$$

$$|\psi(t)\rangle = e^{-\frac{iE_+t}{\hbar}} \cdot \frac{1}{\sqrt{2}}|+z\rangle + e^{-\frac{iE_-t}{\hbar}} \cdot \frac{1}{\sqrt{2}}|-z\rangle = e^{-\frac{i\omega_0 t}{2}} \frac{1}{\sqrt{2}}|+z\rangle + e^{\frac{i\omega_0 t}{2}} \cdot \frac{1}{\sqrt{2}}|-z\rangle$$

$$\langle +z|\psi(t)\rangle = \frac{e^{-\frac{i\omega_0 t}{2}}}{\sqrt{2}} \qquad |\langle +z|\psi(t)\rangle|^2 = \frac{1}{2} \qquad |\langle -z|\psi(t)\rangle|^2 = \frac{1}{2}$$

$$\langle S_z\rangle = \frac{1}{2}\cdot\frac{\hbar}{2} + \frac{1}{2}\left(-\frac{\hbar}{2}\right) = 0$$

$$\langle +x|\psi(t)\rangle = \left(\frac{1}{\sqrt{2}}\langle +z| + \frac{1}{\sqrt{2}}\langle -z|\right)\left(\frac{e^{-\frac{i\omega_0 t}{2}}}{\sqrt{2}}|+z\rangle + \frac{e^{\frac{i\omega_0 t}{2}}}{\sqrt{2}}|-z\rangle\right)$$

$$= \frac{1}{2}\left(e^{-\frac{i\omega_0 t}{2}} + e^{\frac{i\omega_0 t}{2}}\right) = \cos\left(\frac{\omega_0 t}{2}\right)$$

$$|\langle +x|\psi(t)\rangle|^2 = \cos^2\left(\frac{\omega_0 t}{2}\right)$$

$$|\langle -x|\psi(t)\rangle|^2 = 1 - \cos^2\left(\frac{\omega_0 t}{2}\right) = \sin^2\left(\frac{\omega_0 t}{2}\right)$$

$$\langle S_x\rangle = \frac{\hbar}{2}\cos^2\left(\frac{\omega_0 t}{2}\right) + \left(-\frac{\hbar}{2}\right)\sin^2\left(\frac{\omega_0 t}{2}\right) = \frac{\hbar}{2}\left(\cos^2\left(\frac{\omega_0 t}{2}\right) - \sin^2\left(\frac{\omega_0 t}{2}\right)\right) = \frac{\hbar}{2}\cos(\omega_0 t)$$

$$\langle S_y\rangle = \frac{\hbar}{2}\sin(\omega_0 t)$$