$S=1 \qquad m=-1,0,1$

$|1,1\rangle \qquad |1,0\rangle \qquad |1,-1\rangle$

$$\vec{S}_z \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \langle 1,1|\hat{S}_z|1,1\rangle & \langle 1,1|\hat{S}_z|1,0\rangle & \langle 1,1|\hat{S}_z|1,-1\rangle \\ \langle 1,0|\hat{S}_z|1,1\rangle & \langle 1,0|\hat{S}_z|1,0\rangle & \langle 1,0|\hat{S}_z|1,-1\rangle \\ \langle 1,-1|\hat{S}_z|1,1\rangle & \langle 1,-1|\hat{S}_z|1,0\rangle & \langle 1,-1|\hat{S}_z|1,-1\rangle \end{pmatrix} = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\vec{S}_+ \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} \langle 1,1|\hat{S}_+|1,1\rangle & \langle 1,1|\hat{S}_+|1,0\rangle & \langle 1,1|\hat{S}_+|1,-1\rangle \\ \langle 1,0|\hat{S}_+|1,1\rangle & \langle 1,0|\hat{S}_+|1,0\rangle & \langle 1,0|\hat{S}_+|1,-1\rangle \\ \langle 1,-1|\hat{S}_+|1,1\rangle & \langle 1,-1|\hat{S}_+|1,0\rangle & \langle 1,-1|\hat{S}_+|1,-1\rangle \end{pmatrix}$$

$$\hat{S}_+|1,1\rangle = 0 \qquad \hat{S}_+|1,0\rangle = \sqrt{1(1+1)-0(0+1)}\,\hbar|1,1\rangle = \sqrt{2}\,\hbar|1,1\rangle$$

$$\hat{S}_+|1,-1\rangle = \sqrt{1(1+1)-(-1)(-1+1)}\,\hbar|1,0\rangle = \sqrt{2}\,\hbar|1,0\rangle$$

$$\langle 1,0|\hat{S}_+|1,-1\rangle = \sqrt{2}\,\hbar\langle 1,0|1,0\rangle = \sqrt{2}\,\hbar$$

$$\langle 1,1|\hat{S}_+|1,0\rangle = \sqrt{2}\,\hbar\langle 1,1|1,1\rangle = \sqrt{2}\,\hbar$$

$$\vec{S}_+ \xrightarrow[S_z \text{ Basis}]{} \begin{pmatrix} \langle 1,1|\hat{S}_+|1,1\rangle & \langle 1,1|\hat{S}_+|1,0\rangle & \langle 1,1|\hat{S}_+|1,-1\rangle \\ \langle 1,0|\hat{S}_+|1,1\rangle & \langle 1,0|\hat{S}_+|1,0\rangle & \langle 1,0|\hat{S}_+|1,-1\rangle \\ \langle 1,-1|\hat{S}_+|1,1\rangle & \langle 1,-1|\hat{S}_+|1,0\rangle & \langle 1,-1|\hat{S}_+|1,-1\rangle \end{pmatrix} = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{S}_- = \hat{S}_+^+ = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\hat{S}_x = \frac{1}{2}\left(\hat{S}_+ + \hat{S}_-\right)$$

$$\hat{S}_y = \frac{1}{2i}\left(\hat{S}_+ - \hat{S}_-\right)$$

$$\vec{S}_+ \xrightarrow[S_z \text{ basis}]{} \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{S}_- \xrightarrow[S_z \text{ basis}]{} \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\hat{S}_x \xrightarrow[S_z \text{ basis}]{} \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right] = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{S}_y \xrightarrow[S_z \text{ basis}]{} \frac{\hbar}{2i}\left[\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right] = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\hat{S}_x \xrightarrow[S_z \text{ basis}]{} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Finding eigenvalues, eigenvectors:

https://www.youtube.com/watch?v=TQvxWaQnrqI&t

$$\det \left| \begin{array}{ccc|cc} -\lambda & 1 & 0 & -\lambda & 1 \\ 1 & -\lambda & 1 & 1 & -\lambda \\ 0 & 1 & -\lambda & 0 & 1 \end{array} \right| = -\lambda^3 + 0 + 0 - 0 + \lambda + \lambda = 0$$

$$\lambda^3 - 2\lambda = 0$$

$$\lambda(\lambda^2 - 2) = 0$$

$$\lambda = 0, \pm\sqrt{2}$$

Eigenvalues: $0, \pm\hbar$

Finding eigenvectors:

$|1,1\rangle_x$ Eigenvalue: $S_x = \hbar$ $\lambda = \sqrt{2}$

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-\sqrt{2}x_1 + x_2 = 0 \Rightarrow x_1 = \frac{x_2}{\sqrt{2}}$$

$$x_2 - \sqrt{2}x_3 = 0 \Rightarrow x_3 = \frac{x_2}{\sqrt{2}}$$

$$|1,1\rangle_x \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$${}_x\langle 1,1|1,1\rangle_x = 1 = x_2^* (1, \sqrt{2}, 1)\, x_2 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = |x_2|^2 (1+2+1)$$

$$|x_2|^2 = \frac{1}{4} \qquad x_2 = \frac{1}{2}$$

$$|1,1\rangle_x \xrightarrow[S_z \text{ basis}]{} \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$|1,0\rangle_x$ Eigenvalue: $S_x = 0$ $\lambda = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 = 0 \Rightarrow x_2 = 0$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$$|1,0\rangle_x \xrightarrow[S_z \text{ basis}]{} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$${}_x\langle 1,0|1,0\rangle_x = 1 = x_3^* (1,0,1)\, x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = |x_3|^2 (1+1)$$

$|x_3|^2 = \frac{1}{2}$ $x_3 = \frac{1}{\sqrt{2}}$

$$|1,0\rangle_x \xrightarrow[S_z \text{ basis}]{} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$|1,-1\rangle_x \xrightarrow[S_z \text{ basis}]{} \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Time Evolution

$$\hat{U}(t)|\psi(0)\rangle = |\psi(t)\rangle$$

$$\langle\psi(0)|\hat{U}^{+}(t) = \langle\psi(t)|$$

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\underbrace{\hat{U}^{+}(t)\,\hat{U}(t)}_{\hat{1}}|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

$$\hat{U}^{+}(t)\hat{U}(t) = \hat{1}$$

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}\,dt$$

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}dt$$

$$\hat{U}(t+dt) = \hat{U}(dt)\hat{U}(t) = \left(1 - \frac{i}{\hbar}\hat{H}dt\right)\hat{U}(t)$$

$$\hat{U}(t+dt) - \hat{U}(t) = -\frac{idt}{\hbar}\hat{H}\hat{U}(t)$$

$$\frac{\hat{U}(t+dt) - \hat{U}(t)}{dt} = \frac{d}{dt}\hat{U}(t) = -\frac{i}{\hbar}\hat{H}\hat{U}(t)$$

$$i\hbar\frac{d}{dt}\hat{U}(t)|\psi(0)\rangle = \hat{H}\hat{U}(t)|\psi(0)\rangle$$

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}dt$$

$$\hat{U}(t) = \lim_{N\to\infty}\left(1 - \frac{i}{\hbar}\hat{H}\frac{t}{N}\right)^N = e^{-\frac{i\hat{H}t}{\hbar}}$$

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$$

$$\langle\psi(t)|\hat{H}|\psi(t)\rangle = \langle\psi(0)|\hat{U}^{+}(t)\hat{H}\hat{U}(t)|\psi(0)\rangle = \langle\psi(0)|\hat{H}\hat{U}^{+}(t)\hat{U}(t)|\psi(0)\rangle = \langle\psi(0)|\hat{H}|\psi(0)\rangle$$

Erwin Schrödinger