$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \qquad [\hat{J}_z, \hat{J}^2] = 0$$

$|\lambda, m\rangle$ shared eigenstate of $\hat{J}_z$ and $\hat{J}^2$

$$\hat{J}^2|\lambda, m\rangle = \lambda\hbar^2|\lambda, m\rangle \qquad \hat{J}_z|\lambda, m\rangle = m\hbar|\lambda, m\rangle$$

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \qquad \hat{J}_+|\lambda, m\rangle = c_+\hbar|\lambda, m+1\rangle$$

$$\hat{J}_- = \hat{J}_x - i\hat{J}_y \qquad \hat{J}_-|\lambda, m\rangle = c_-\hbar|\lambda, m-1\rangle$$

Assume j is max m: $\hat{J}_+|\lambda, j\rangle = 0$ $\qquad j' = -j$

Assume j' is min m: $\hat{J}_-|\lambda, j'\rangle = 0$ $\qquad \lambda = j^2 + j = j(j+1)$

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots \qquad m = -j, -j+1, -j+2, \ldots, j-1, j$$

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$j = 0, 1, 2, 3 \ldots$$

Bosons

$$j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots$$

Fermions

$$|\lambda, m\rangle \longrightarrow |j, m\rangle$$

$$\hat{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$$

$$\hat{J}_z |j, m\rangle = \hbar m |j, m\rangle$$

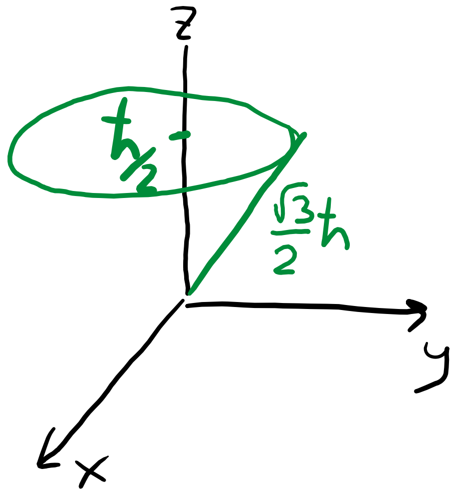
Examples:

$j = \frac{1}{2}$: $m = -\frac{1}{2}, \frac{1}{2}$

$|\frac{1}{2}, -\frac{1}{2}\rangle = |-z\rangle$ $|\frac{1}{2}, \frac{1}{2}\rangle = |+z\rangle$

$\hat{J}_z |\frac{1}{2}, -\frac{1}{2}\rangle = -\frac{1}{2}\hbar |\frac{1}{2}, -\frac{1}{2}\rangle$ $\hat{J}_z |\frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{2}\hbar |\frac{1}{2}, \frac{1}{2}\rangle$

$\hat{J}^2 |\frac{1}{2}, -\frac{1}{2}\rangle = \hbar^2 \frac{1}{2} \cdot \left(\frac{1}{2}+1\right) |\frac{1}{2}, -\frac{1}{2}\rangle = \frac{3\hbar^2}{4} |\frac{1}{2}, -\frac{1}{2}\rangle$



$j = 1$ $\quad m = -1, 0, 1$

$|1,-1\rangle$ $\quad |1,0\rangle$ $\quad |1,1\rangle$

$\hat{J}_z|1,-1\rangle = -\hbar|1,-1\rangle$

$\hat{J}_z|1,0\rangle = 0$

$\hat{J}_z|1,1\rangle = \hbar|1,1\rangle$

$$\hat{J}^2|1,0\rangle = 1(1+1)\hbar^2|1,0\rangle = 2\hbar^2|1,0\rangle$$

$j = \frac{3}{2}$ $\qquad$ $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$\left|\frac{3}{2}, -\frac{3}{2}\right\rangle \qquad \left|\frac{3}{2}, -\frac{1}{2}\right\rangle \qquad \left|\frac{3}{2}, \frac{1}{2}\right\rangle \qquad \left|\frac{3}{2}, \frac{3}{2}\right\rangle$

$\hat{J}_z\left|\frac{3}{2}, -\frac{3}{2}\right\rangle = -\frac{3}{2}\hbar\left|\frac{3}{2}, \frac{3}{2}\right\rangle \qquad \hat{J}_z\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = -\frac{1}{2}\hbar\left|\frac{3}{2}, -\frac{1}{2}\right\rangle \qquad \hat{J}_z\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{2}\hbar\left|\frac{3}{2}, \frac{1}{2}\right\rangle \qquad \hat{J}_z\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \frac{3}{2}\hbar\left|\frac{3}{2}, \frac{3}{2}\right\rangle$

$$\hat{J}^2\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{15}{4}\hbar^2\left|\frac{3}{2}, -\frac{1}{2}\right\rangle$$

$$\sqrt{\frac{15}{4}\hbar^2} = \frac{\sqrt{15}\,\hbar}{2} > \frac{3}{2}\hbar$$

$$\hat{J}_+ |j,m\rangle = c_+ \hbar |j,m+1\rangle$$

$$\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z$$

$$\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hbar \hat{J}_z$$

$$\langle j,m| \hat{J}_+^{+} = c_+^* \hbar \langle j, m+1|$$

$$\langle j,m| \hat{J}_- \hat{J}_+ |j,m\rangle = c_+^* c_+ \hbar^2 \underbrace{\langle j,m+1|j,m+1\rangle}_{1}$$

$$\langle j,m| \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z |j,m\rangle = \langle j,m| j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2 |j,m\rangle = \big(j(j+1) - m(m+1)\big)\hbar^2 \langle j,m|j,m\rangle$$

$$\big(j(j+1) - m(m+1)\big)\hbar^2 = |c_+|^2 \hbar^2$$

$$c_+ = \sqrt{j(j+1) - m(m+1)}$$

$$\hat{J}_+ |j,m\rangle = \sqrt{j(j+1) - m(m+1)}\, \hbar\, |j, m+1\rangle$$

$$\hat{J}_-|j,m\rangle = c_-\hbar|j,m-1\rangle$$

$$\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$$

$$\langle j,m|\hat{J}_-^+ = c_-^*\hbar\langle j,m-1|$$

$$\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$$

$$\langle j,m|\hat{J}_+\hat{J}_-|j,m\rangle = c_-^*c_-\hbar^2\langle j,m-1|j,m-1\rangle$$

$$\langle j,m|\hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z|j,m\rangle = \langle j,m|j(j+1)\hbar^2 - m^2\hbar^2 + m\hbar^2|j,m\rangle = \big(j(j+1) - m(m-1)\big)\hbar^2\langle j,m|j,m\rangle$$

$$\big(j(j+1) - m(m-1)\big)\hbar^2 = |c_-|^2\hbar^2$$

$$c_- = \sqrt{j(j+1) - m(m-1)}$$

$$\hat{J}_-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar|j,m-1\rangle$$

Example Spin-3 particle

$$|\psi\rangle = \frac{1}{\sqrt{2}}|3,1\rangle + \frac{1}{\sqrt{2}}|3,2\rangle$$

$\langle J_z\rangle$ - ? $\langle J_x\rangle$ - ?

$\langle J_z^2\rangle$ - ? $\langle J_x^2\rangle$ - ?

ΔJ_z - ? ΔJ_x - ?

$$\hat{J}_z|\psi\rangle = \frac{1}{\sqrt{2}}\hat{J}_z|3,1\rangle + \frac{1}{\sqrt{2}}\hat{J}_z|3,2\rangle = \frac{1}{\sqrt{2}}\hbar|3,1\rangle + \frac{1}{\sqrt{2}}2\hbar|3,2\rangle$$

$$\langle J_z\rangle = \langle\psi|\hat{J}_z|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle 3,1| + \frac{1}{\sqrt{2}}\langle 3,2|\right)\left(\frac{\hbar}{\sqrt{2}}|3,1\rangle + \frac{2\hbar}{\sqrt{2}}|3,2\rangle\right) = \frac{\hbar}{2} + \frac{2\hbar}{2} = \frac{3\hbar}{2}$$

$$\langle J_z^2\rangle = \langle\psi|\hat{J}_z\hat{J}_z|\psi\rangle = \left(\frac{\hbar}{\sqrt{2}}\langle 3,1| + \frac{2\hbar}{\sqrt{2}}\langle 3,2|\right)\left(\frac{\hbar}{\sqrt{2}}|3,1\rangle + \frac{2\hbar}{\sqrt{2}}|3,2\rangle\right) = \frac{\hbar^2}{2} + \frac{4\hbar^2}{2} = \frac{5\hbar^2}{2}$$

$$\Delta J_z = \sqrt{\langle J_z^2\rangle - \langle J_z\rangle^2} = \sqrt{\frac{5\hbar^2}{2} - \left(\frac{3\hbar}{2}\right)^2} = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2}$$

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y$$
$$\hat{J}_- = \hat{J}_x - i\hat{J}_y$$
$$\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$$

$$\hat{J}_+|j,m\rangle = \sqrt{j(j+1)-m(m+1)}\,\hbar\,|j,m+1\rangle$$
$$\hat{J}_-|j,m\rangle = \sqrt{j(j+1)-m(m-1)}\,\hbar\,|j,m-1\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|3,1\rangle + \frac{1}{\sqrt{2}}|3,2\rangle$$

$$\hat{J}_x|\psi\rangle = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)\left(\frac{1}{\sqrt{2}}|3,1\rangle + \frac{1}{\sqrt{2}}|3,2\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\sqrt{3(3+1)-1(1+1)}\,\hbar|3,2\rangle + \sqrt{3(3+1)-2(2+1)}\,\hbar|3,3\rangle + \sqrt{3(3+1)-1(1-1)}\,\hbar|3,0\rangle + \sqrt{3(3+1)-2(2-1)}\,\hbar|3,1\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\sqrt{10}\,\hbar|3,2\rangle + \sqrt{6}\,\hbar|3,3\rangle + \sqrt{12}\,\hbar|3,0\rangle + \sqrt{10}\,\hbar|3,1\rangle\right)$$

$$\langle\hat{J}_x\rangle = \langle\psi|\hat{J}_x|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle 3,1| + \frac{1}{\sqrt{2}}\langle 3,2|\right)\frac{1}{2\sqrt{2}}\left(\sqrt{10}\,\hbar|3,2\rangle + \sqrt{6}\,\hbar|3,3\rangle + \sqrt{12}\,\hbar|3,0\rangle + \sqrt{10}\,\hbar|3,1\rangle\right)$$

$$= \frac{\sqrt{10}}{4}\hbar + \frac{\sqrt{10}}{4}\hbar = \frac{\sqrt{10}}{2}\hbar$$

$$\hat{J}_x|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{10}\,\hbar|3,2\rangle + \sqrt{6}\,\hbar|3,3\rangle + \sqrt{12}\,\hbar|3,0\rangle + \sqrt{10}\,\hbar|3,1\rangle\right)$$

$$\langle J_x^2\rangle = \langle\psi|\hat{J}_x\hat{J}_x|\psi\rangle =$$

$$\frac{1}{2\sqrt{2}}\left(\sqrt{10}\hbar\langle 3,2| + \sqrt{6}\hbar\langle 3,3| + \sqrt{12}\hbar\langle 3,0| + \sqrt{10}\hbar\langle 3,1|\right)\frac{1}{2\sqrt{2}}\left(\sqrt{10}\hbar|3,2\rangle + \sqrt{6}\hbar|3,3\rangle + \sqrt{12}\hbar|3,0\rangle + \sqrt{10}\hbar|3,1\rangle\right)$$

$$= \left(\frac{1}{2\sqrt{2}}\right)^2\left(10\hbar^2 + 6\hbar^2 + 12\hbar^2 + 10\hbar^2\right) = \frac{1}{8}(38\hbar^2) = \frac{19}{4}\hbar^2$$

$$\Delta J_x = \sqrt{\langle J_x^2\rangle - \langle J_x\rangle^2} = \sqrt{\frac{19}{4}\hbar^2 - \left(\frac{\sqrt{10}}{2}\hbar\right)^2} = \frac{3}{2}\hbar$$

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

Given: $[\hat{A},\hat{B}]=i\hat{C}$

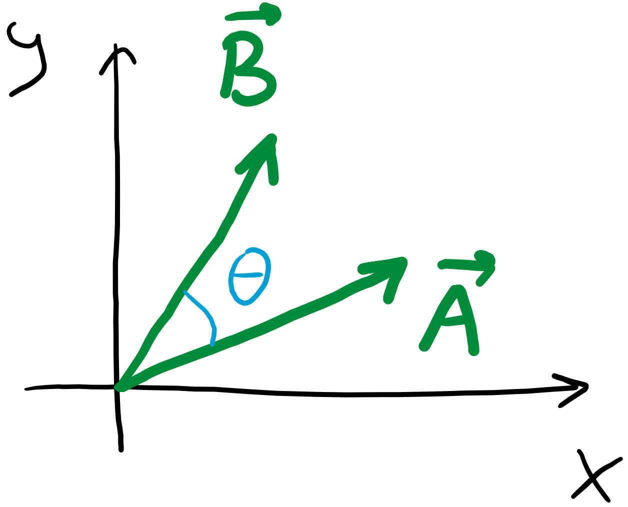
$$|\alpha\rangle = (\hat{A}-\langle A\rangle)|\psi\rangle$$

$$|\beta\rangle = (\hat{B}-\langle B\rangle)|\psi\rangle$$

$$\langle\alpha|\alpha\rangle = \langle\psi|(\hat{A}-\langle A\rangle)(\hat{A}-\langle A\rangle)|\psi\rangle$$

$$= \langle\psi|(\hat{A}-\langle A\rangle)^2|\psi\rangle = \langle(\hat{A}-\langle A\rangle)^2\rangle = (\Delta A)^2$$

$$\langle\beta|\beta\rangle = \langle\psi|(\hat{B}-\langle B\rangle)(\hat{B}-\langle B\rangle)|\psi\rangle = (\Delta B)^2$$



$$\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cdot\cos\theta$$

$$|\vec{A}\cdot\vec{B}| \leq |\vec{A}|\cdot|\vec{B}|$$

$$|\vec{A}|^2\cdot|\vec{B}|^2 \geq |\vec{A}\cdot\vec{B}|^2$$

Schwarz inequality

$$\langle\alpha|\beta\rangle = \langle\psi|(\hat{A}-\langle A\rangle)(\hat{B}-\langle B\rangle)|\psi\rangle$$

$$= \langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle B\rangle\langle\psi|\hat{A}|\psi\rangle - \langle A\rangle\langle\psi|\hat{B}|\psi\rangle + \langle A\rangle\langle B\rangle\langle\psi|\psi\rangle$$

$$= \langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle A\rangle\langle B\rangle$$

$$z - z^* = 2i\,\mathrm{Im}(z)$$

$$\hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C}$$

$$\langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle\psi|\hat{B}\hat{A}|\psi\rangle = i\langle\psi|\hat{C}|\psi\rangle$$

$$\mathrm{Im}(\langle AB\rangle) = \frac{\langle C\rangle}{2}$$

$$2i\,\mathrm{Im}(\langle\psi|\hat{A}\hat{B}|\psi\rangle) = i\langle\psi|\hat{C}|\psi\rangle$$

$$|\langle\alpha|\beta\rangle| = |\langle AB\rangle - \langle A\rangle\langle B\rangle| \geq |\mathrm{Im}(\langle AB\rangle)| = \left|\frac{\langle C\rangle}{2}\right|$$

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

$$(\Delta A)^2\cdot(\Delta B)^2 \geq |\langle\alpha|\beta\rangle|^2 \geq \left|\frac{\langle C\rangle}{2}\right|^2$$

$$\Delta A\cdot\Delta B \geq \left|\frac{\langle C\rangle}{2}\right|$$

$$[\hat{A}, \hat{B}] = i\hat{C} \quad \Rightarrow \quad \Delta A \cdot \Delta B \geq \left| \frac{\langle C \rangle}{2} \right|$$

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \quad \Rightarrow \quad \Delta J_x \cdot \Delta J_y \geq \frac{\hbar}{2} |\langle J_z \rangle|$$

$$\Delta J_z \Delta J_x \geq \frac{\hbar}{2} |\langle J_y \rangle|$$

$$\Delta J_y \cdot \Delta J_z \geq \frac{\hbar}{2} |\langle J_x \rangle|$$

Example

$$|\psi\rangle = |+x\rangle$$

$$\hat{J}_x |+x\rangle = \frac{\hbar}{2} |+x\rangle$$

$$\langle +x| \hat{J}_x |+x\rangle = \frac{\hbar}{2}$$

$$\langle +x| \hat{J}_x^2 |+x\rangle = \frac{\hbar^2}{4}$$

$$(\Delta J_x)^2 = \langle J_x^2\rangle - \langle J_x\rangle^2 = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2 = 0$$

$$\langle \hat{J}_z\rangle = \langle +x| \hat{J}_z |+x\rangle$$

$$\hat{J}_z |+x\rangle = \hat{J}_z \left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right) = \frac{1}{\sqrt{2}}\frac{\hbar}{2}|+z\rangle + \frac{1}{\sqrt{2}}\left(-\frac{\hbar}{2}\right)|-z\rangle = \frac{\hbar}{2}|-x\rangle$$

$$\langle J_z\rangle = \langle +x| \hat{J}_z |+x\rangle = \frac{\hbar}{2}\langle +x|-x\rangle = 0$$

$$\Delta J_x \cdot \Delta J_y = 0 = \langle J_z\rangle$$

$$[\hat{X}, \hat{p}_x] = i\underset{\hat{C}}{\hbar} \quad \Rightarrow \quad \underbrace{\Delta X \cdot \Delta p_x \geq \frac{1}{2}\hbar}$$

$$[\hat{A}, \hat{B}] = i\hat{C} \quad \Rightarrow \quad \Delta A \cdot \Delta B \geq \frac{1}{2}|\langle C \rangle|$$